INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES

THE TOPOLOGY OF NORMAL SINGULARITIES OF AN ALGEBRAIC SURFACE AND A CRITERION FOR SIMPLICITY

by David MUMFORD

CHARACTERS AND COHOMOLOGY OF FINITE GROUPS

by M. F. ATIYAH

1961

BLICATIONS MATHÉMATIQUES, N° 9

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THE TOPOLOGY OF NORMAL SINGULARITIES OF AN ALGEBRAIC SURFACE AND A CRITERION FOR SIMPLICITY

By David MUMFORD

Let a variety V^n be embedded in complex projective space of dimension m. Let $P \in V$. About P, choose a ball U of small radius ε, in some affine metric $ds^2 = \Sigma dx_j^2 + \Sigma dy_j^2$, $z_j = x_j + iy_j$ affine coordinates. Let B be its boundary and $M = B \cap V$. Then M is a real complex of dimension $2n-1$, and a manifold if P is an isolated singularity. The topology of M together with its embedding in B ($= a\ 2m-1$-sphere) reflects the nature of the point P in V. The simplest case and the only one to be studied so far, to the author's knowledge, is where $n=1, m=2$, i.e. a plane curve (see [3], [14]). Then M is a disjoint union of a finite number of circles, knotted and linked in a 3-sphere. There is one circle for each branch of V at P, the intersection number of each pair of branches is the linking number of the corresponding circles, and the knots formed by each circle are compound toroidal, their canonical decomposition reflecting exactly the decomposition of each branch via infinitely near points.

The next interesting case is $n=2, m=3$. One would hope to find knots of a 3-sphere in a 5-sphere in this case; this would come about if P were an isolated singularity whose normalization was non-singular. Unfortunately, isolated non-normal points do not occur on hyper-surfaces in any Cohen-MacCaulay varieties. What happens, however, if the normalization of P is non-singular, is that M is the image of a 3-sphere mapped into a 5-sphere by a map which (i) identifies several circles, and (ii) annihilates a ray of tangent vectors at every point of another set of circles. In many cases the second does not occur, and we have an immersion of the 3-sphere in the 5-sphere. It would be quite interesting to know Smale's invariant in $\pi_3(V_{3,5})$ in this case (see [10]).

From the standpoint of the theory of algebraic surfaces, the really interesting case is that of a singular point on a *normal* algebraic surface, and m arbitrary. M is then by no means generally S^3 and consequently its own topology reflects the singularity P! In this paper, we shall consider this case, first giving a partial construction of $\pi_1(M)$ in terms of a resolution of the singular point P; secondly we shall sketch the connexion between $H_1(M)$ and the algebraic nature of P. Finally and principally, we shall demonstrate the following theorem, conjectured by Abhyankar:

Theorem. — $\pi_1(M) = (e)$ if and only if P is a simple point of F (a locally normal surface); and F topologically a manifold at P implies $\pi_1(M) = (e)$.

1. — ANALYSIS OF M AND PARTIAL CALCULATION OF $\pi_1(M)$

A normal point P in F is given. A finite sequence of quadratic transformations plus normalizations leads to a non-singular surface F′ dominating F [15]. The inverse image of P on F′ is the union of a finite set of curves $E_1, E_2, \ldots, E_n$. By further quadratic transformations if necessary we may assume that all E_i are non-singular, and, if $i \neq j$, and $E_i \cap E_j \neq \emptyset$, then that E_i and E_j intersect normally in exactly one point, which does not lie on any other E_k. This will be a great technical convenience.

We note at this point the following fundamental fact about E_i : the intersection matrix $S = ((E_i . E_j))$ is negative definite. (This could also be proven by Hodge's Index Theorem.)

Proof. — Let H_1 and H_2 be two hyperplane sections of F, H_1 through P, and H_2 not (and also not through any other singular points of F). Let $(f) = H_1 - H_2$. Let H_1' be the proper transform of H_1 on F′, and H_2' the total transform of H_2. Then $H_2' \equiv H_1' + \Sigma m_i E_i$, where $m_i > 0$, all i (here m_i is positive since $m_i = \mathrm{ord}_{E_i}(f)$, f a function that is regular and zero at P on F, and moreover P is the center of the valuation of E_i on F).

Let $S' = ((m_i E_i . m_j E_j)) = M.S.M$, where M is the diagonal matrix with $M_{ii} = m_i$. To prove S′ is negative definite is equivalent with the desired assertion. Now note (*a*) $S_{ij} \geq 0$, if $i \neq j$, (*b*) $\sum\limits_i S'_{ij} = \sum\limits_i (m_i E_i . m_j E_j) = -(H_1' . m_j E_j) \leq 0$, all j. For any symmetric matrix S′, these two facts imply negative indefiniteness. To get definiteness, look closer: we know also (*c*) $\sum\limits_i S'_{ij} < 0$, for some j (since H_1' passes through some E_j), and (*d*) we cannot split $(1, 2, \ldots, n) = (i_1, i_2, \ldots, i_k) \cup (j_1, j_2, \ldots, j_{n-k})$ disjointly so that $S'_{i_a j_b} = 0$, any a, b (since $\bigcup E_i$ is connected by Zariski's main theorem [16]). Now these together give definiteness: Say

$$0 = \sum_{ij} \alpha_i \alpha_j S'_{ij} = \Sigma \alpha_i^2 S'_{ii} + 2 \sum_{i<j} \alpha_i \alpha_j S'_{ij}$$
$$= \sum_j \Big(\sum_i S'_{ij}\Big) \alpha_j^2 - \sum_{i<j} S'_{ij} (\alpha_i - \alpha_j)^2$$

where α_i are real. Then by (*c*), some $\alpha_j = 0$, and by (*d*), $\alpha_i = \alpha_j$, all i, j.

Our first step is a close analysis of the structure of M. We have defined it informally in the introduction in terms of an affine metric (depending apparently on the choice of this metric). Here we shall give a more general definition, and show that all these manifolds coincide, by virtue of having identical constructions by patching maps.

In the introduction, M is a level manifold of the positive C^∞ fcn.

$$p^2 = |Z_1|^2 + \ldots + |Z_n|^2,$$

(Z_i affine coordinates near $P \in F$). Now notice that M may also be defined as the level manifolds of p^2 on the non-singular F′ (p^2 being canonically identified to a fcn. on F′). It is as a "tubular neighborhood" of $\bigcup E_i \subset F'$ that we wish to discuss M. Now the general problem, given a complex $K \subset E^n$, Euclidean n-space, to define a tubular neighborhood,

has been attacked by topologists in several ways although it does not appear to have been treated definitively as yet. J. H. C. Whitehead [13], when K is a subcomplex in a triangulation of E^n, has defined it as the boundary of the star of K in the second barycentric subdivision of the given triangulation. I am informed that Thom [11] has considered it more from our point of view: for a suitably restricted class of positive C^∞ fcns. f such that $f(P)=0$ if and only if $P\in K$, define the tubular neighborhood of K to be the level manifolds $f=\varepsilon$, small ε. The catch is how to suitably restrict f; here the archtype for f^{-1} may be thought of as the potential distribution due to a uniform charge on K. In our case, as we have no wish to find the topological ultimate, we shall merely formulate a convenient, and convincingly broad class of such f, which includes the p^2 of the introduction.

Let us say that a positive C^∞ real fcn. f on F' such that $f(P)=0$ iff $P\in E_i$, is *admissible* if

1) $\forall P\in E_i - \bigcup_{j\neq i} E_j$, if $Z=0$ is a local equation for E_i near P, $f=|Z|^{2n_i}.g$, where g is C^∞ and neither 0 nor ∞ near P.

2) If $P_{ij}=E_i\cap E_j$, and $Z=0$, $W=0$ are local equations for E_i, E_j respectively then $f=|Z|^{2n_i}.|W|^{2n_j}.g$, where g is C^∞ and neither 0 nor ∞ near P_{ij}.

The following proposition is left to the reader.

Proposition: (i) If F'' dominates F', and f is admissible for $\bigcup E_i$ on F', and $g: F''\to F'$ is the canonical map, then $f\circ g$ is admissible for $g^{-1}(\bigcup E_i)$ on F'.

(ii) For a suitable F'' dominating F', p^2 is an admissible map for $g^{-1}(\bigcup E_i)$.

Let me say, however, that in (ii), the point is to take F'' high enough so that the linear system of zeroes of the functions $(\Sigma\alpha_i Z_i)$ less its fixed components, has no base points.

What we must now show is that there is a unique manifold M such that, if f is any admissible fcn., M is homeomorphic to $\{P\,|\,f(P)=\varepsilon\}$ for all sufficiently small ε. Fix a fcn. f to be considered. Notice that at each of the points P_{ij}, there exist real C^∞ coordinates X_{ij}, Y_{ij}, U_{ij}, V_{ij}, such that

$$f=(X_{ij}^2+Y_{ij}^2)^{n_i}\,(U_{ij}^2+V_{ij}^2)^{n_j}\,\alpha_{ij},$$

α_{ij} *a constant*, valid in some neighborhood U given by

$$\begin{aligned}X_{ij}^2+Y_{ij}^2&<1\\ U_{ij}^2+V_{ij}^2&<1.\end{aligned}$$

Assume E_i is $X_{ij}=Y_{ij}=0$, and E_j is $U_{ij}=V_{ij}=0$.

Our first trick consists of choosing a C^∞ metric $(ds)^2$ (depending on f), such that within

$$U'=\begin{Bmatrix}X_{ij}^2+Y_{ij}^2<1/2\\ U_{ij}^2+V_{ij}^2<1/2\end{Bmatrix},$$

$$ds^2=dX_{ij}^2+dY_{ij}^2+dU_{ij}^2+dV_{ij}^2.$$

Such a metric exists, e.g. by averaging a Hodge metric with these Euclidean metrics by some partition of unity. Now let

$$\begin{array}{ccc} N_i & & S_i \\ \downarrow{\scriptstyle \pi_i} & \text{and} & \downarrow{\scriptstyle \psi_i} \\ E_i & & E_i \end{array}$$

be the normal 2-plane bundle to E_i and normal S^1-bundle to E_i in F' respectively. Consider the map $(\exp)_i$: $N_i \to F'$ obtained by mapping N_i into F along geodesics perpendicular to E_i. Let $f_i = f \circ (\exp)_i$. Now for every point $Q \in E_i - \bigcup_{j \neq i} E_j$, there is a neighborhood W of $Q \in E_i$, and an ε_0 such that if $\varepsilon < \varepsilon_0$, the locus $f_i(P) = \varepsilon$, $\pi_i(P) \in W$ cuts once each ray in $\pi_i^{-1}(W)$ (because f_i^{1/n_i} is a well-defined pos. C^∞ fcn. vanishing on the zero cross-section, with non-degenerate Hessian in normal directions; this is the standard situation of Morse theory, see [9]). Consequently, for any $W \subset E_i$ open, such that $E_j \cap W = \emptyset$, $j \neq i$, there is an ε_0 such that if $\varepsilon < \varepsilon_0$, the locus $f(P) = \varepsilon$ canonically contains a homeomorphic image of $\psi_i^{-1}(W)$ (recall $(\exp)_i$ is a local homeomorphism near the zero-section of N_i). Therefore, we see that the manifold M for which we are seeking a definition independent of f, is to be put together out of pieces of S_i; we need only seek its structure near P_{ij}. Let us therefore look in U'. Let us fix neighborhoods U_{ij} of $P_{ij} \in E_i$ and U_{ji} of $P_{ij} \in E_j$ by $(U_{ij}^2 + V_{ij}^2) < 1/4$ and $(X_{ij}^2 + Y_{ij}^2) < 1/4$ respectively. Let $E_k^* = E_k - \bigcup_{j \neq k} U_{kj}$ for all k. Now choose $\varepsilon_0 < \alpha_{i,j}/8^{n_i+n_j}$ and so that if $\varepsilon < \varepsilon_0$, $f(P) = \varepsilon$ contains $\psi_i^{-1}(E_i^*)$ and $\psi_j^{-1}(E_j^*)$ canonically. Then in the local coordinates in U' about P_{ij}, $\psi_i^{-1}(\partial E_i^*) \subset \{P \mid f(P) = \varepsilon\}$ equals

$$\left\{ (X_{ij}, Y_{ij}, U_{ij}, V_{ij}) \mid U_{ij}^2 + V_{ij}^2 = 1/4,\ X_{ij}^2 + Y_{ij}^2 = \left(\frac{4^{n_j}\varepsilon}{\alpha_{ij}}\right)^{1/n_i} \right\}$$

and $\psi_j^{-1}(\partial E_j^*) \subset \{P \mid f(P) = \varepsilon\}$ equals

$$\left\{ (X_{ij}, Y_{ij}, U_{ij}, V_{ij}) \mid X_{ij}^2 + Y_{ij}^2 = 1/4,\ U_{ij}^2 + V_{ij}^2 = \left(\frac{4^{n_i}\varepsilon}{\alpha_{ij}}\right)^{1/n_j} \right\}$$

(because of the Euclidean character of the metric ds^2 near P_{ij}, $\exp_i$ takes the simplest possible form!). Note $\left(\frac{4^{n_j}\varepsilon}{\alpha_{ij}}\right)^{1/n_i} < 1/8$. Therefore, we see that $\psi_i^{-1}(E_i^*)$ and $\psi_j^{-1}(E_j^*)$ are patched by a standard "plumbing fixture":

$$\{(x, y, u, v) \mid (x^2 + y^2) \leqslant 1/4,\ (u^2 + v^2) \leqslant 1/4,\ (x^2 + y^2)^n . (u^2 + v^2)^m = \varepsilon < 1/8^{n+m}\}$$

where n and m are integers.

One sees immediately that this is simply $S^1 \times S^1 \times [0, 1]$, and if we set $M_i^* = \psi_i^{-1}(E_i^*)$, then it simply attaches ∂M_i^* to ∂M_j^*. Moreover, what is this attaching? There is a coordinate system on both ∂M_i^* and ∂M_j^* via

$$\left(\frac{X_{ij}}{\sqrt{X_{ij}^2+Y_{ij}^2}}, \frac{Y_{ij}}{\sqrt{X_{ij}^2+Y_{ij}^2}}\right)=\xi\in S^1 \text{ (in the usual embedding in } E^2)$$

$$\left(\frac{U_{ij}}{\sqrt{U_{ij}^2+V_{ij}^2}}, \frac{V_{ij}}{\sqrt{U_{ij}^2+V_{ij}^2}}\right)=\eta\in S^1 \text{ (in the usual embedding in } E^2)$$

and relative to these coordinates, the attaching is readily seen to be the identity. To complete the invariant topological description of M, we need only to show that the cycles $\{(\xi, \eta_0) \mid \xi\in S^1, \eta_0 \text{ fixed}\}$ and $\{(\xi_0, \eta) \mid \xi_0 \text{ fixed}, \eta\in S^1\}$ are invariantly determined (since an identification of 2 tori is determined up to isotopy by an identification of a basis of 1-cycles). But on M_i^* for instance, the 1st one is just the fibre of S_i over a point of E_i, and the 2nd is the loop ∂E_i^* lifted to S_i so that it is contractible in $\psi_i^{-1}(U_{ij})$; similarly on M_j^*, but *vice versa*.

This determines M uniquely. We have essentially found, moreover, not only M but also for any fixed f, maps

$$\varphi : M \to \cup E_i$$
$$\psi : \{P \mid 0 < f(P) \leqslant \varepsilon\} \to M$$

where ψ induces a homeomorphism of any $\{P \mid f(P)=\varepsilon' \leqslant \varepsilon\}$ onto M. Namely, define φ on M_i^* by ψ_i: projection into E_i, and in U′ near P_{ij}, define it as follows (fig. 1):

$$\begin{aligned}\varphi((X_{ij}, Y_{ij}, U_{ij}, V_{ij})) &= (0, 0, U_{ij}, V_{ij})\in E_i \quad \text{if } U_{ij}^2+V_{ij}^2\geq 1/4\\ &= (0, 0, pU_{ij}, pV_{ij})\in E_i \quad \text{if } X_{ij}^2+Y_{ij}^2\leqslant U_{ij}^2+V_{ij}^2\leqslant 1/4\\ &= (p'X_{ij}, p'Y_{ij}, 0, 0)\in E_j \quad \text{if } U_{ij}^2+V_{ij}^2\leqslant X_{ij}^2+Y_{ij}^2\leqslant 1/4\\ &= (X_{ij}, Y_{ij}, 0, 0)\in E_j \quad \text{if } X_{ij}^2+Y_{ij}^2\geqslant 1/4,\end{aligned}$$

where

$$p=\tau(X_{ij}^2+Y_{ij}^2, U_{ij}^2+V_{ij}^2)$$
$$p'=\tau(U_{ij}^2+V_{ij}^2, X_{ij}^2+Y_{ij}^2)$$

and where

$$\tau(\alpha, \beta) = \frac{\beta-\alpha}{1-4\alpha}.$$

As for ψ, away from P_{ij}, define ψ by first $(\exp)_i^{-1}$, then the projection of N_i—(0-section) to S_i, and then the identification of S_i into M; near P_{ij}, define it by identifying those points whose ξ and η coordinates are equal, and that have the same image in $E_i\cup E_j$ under the map φ.

Note that φ induces a map $\varphi_*: \pi_1(M)\to\pi_1(\cup E_i)$, which is onto as all the "fibres" are connected [1]. In order not to be lost in a morass of confusion, we shall now restrict ourselves to computing only H_1 in general, and π_1 only if $\pi_1(\cup E_i)=(e)$. Note thats this last is equivalent to (*a*) E_i connected together as a tree (i.e. it never happens $E_1\cap E_2\neq\varnothing$, $E_2\cap E_3\neq\varnothing$, $\ldots$, $E_{k-1}\cap E_k\neq\varnothing$, $E_k\cap E_1\neq\varnothing$ and $k>2$ for some ordering of the E_i's), (*b*) all E_i are rational curves.

First, to compute $H_1(M)$, start with $H_i(\cup E_i)$. Let $\cup E_i$, as a graph, be p-connected,

[1] M is, of course, not a fibre space in the usual sense. However, the map φ_* in question is onto for any simplicial map such that the inverse image of every point is connected.

i.e. there exist some $P_1, \ldots, P_p$ such that if these points are deleted from $\cup E_i$, then $\cup E_i$ becomes a tree, but this does not happen for fewer P_i. Choose such P_i, and to $\cup E_i - \cup P_i$, for each P_i, add two points P_i' and P_i'', one to each E_j to which P_i belonged. The result, T, is, up to homotopy type, simply the wedge of the (closed) surfaces E_i [1]. $\cup E_i$ is itself obtained from T by identifying the p pairs of points P_i', P_i''; therefore up to homotopy

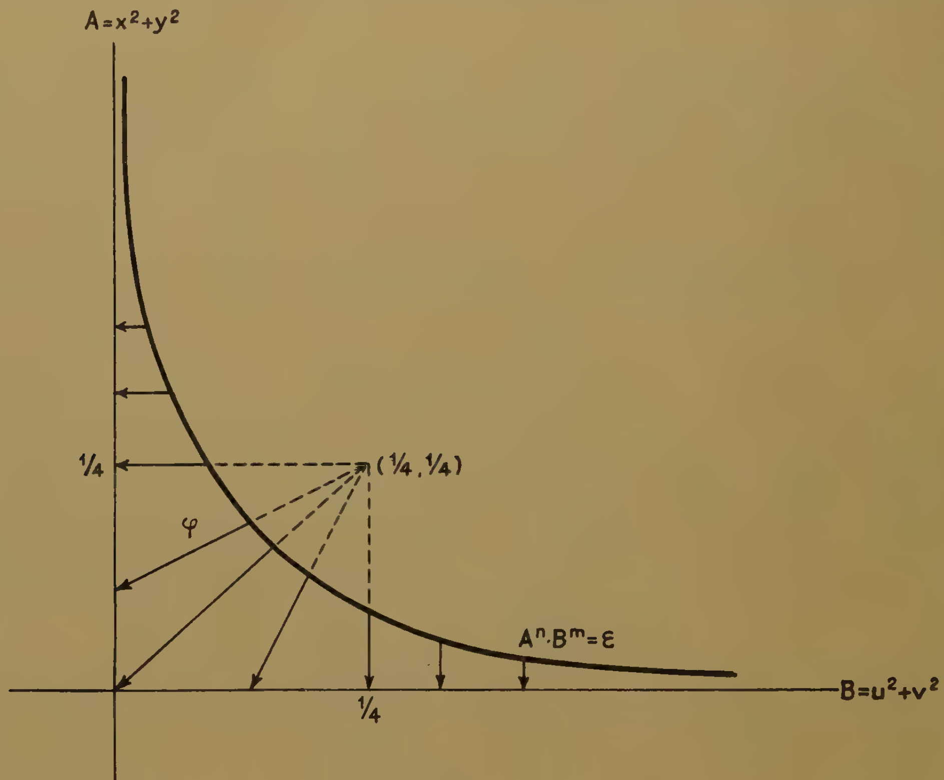

Fig. 1

type, it is the wedge of E_i and p loops. Therefore $H_1(\cup E_i) = \mathbf{Z}^{p+2\Sigma g_i}$, where g_i is the genus of E_i.

Now φ_* induces an onto map $H_1(M) \to H_1(\cup E_i)$, by passing modulo the commutators. Let K be its kernel. Let α_i be the loop or cycle of M consisting of the fibre of M over some point in $E_i - \cup_{j \neq i} E_j$ with the following sense: if $f_i = 0$ is a local equation for E_i,

$$\int_{\alpha_i} \frac{df_i}{f_i} = +2\pi i$$

or equivalently α_i as a loop about the origin of a fibre of the normal bundle N_i to E_i should have positive sense in its canonical orientation. I claim α_i generate K, and their relations are exactly $\Sigma(E_i . E_j)\alpha_j = 0$, $i = 1, \ldots, n$.

(1) For example, proceeding surface by surface in any order, we may deform the complex $\cup E_i$ so that all the E_j which meet some one E_i meet it at the same point.

Proof. — First introduce the auxiliary cycles β_{ij} on $\varphi^{-1}(E_i)=M_i$, whenever $E_i \cap E_j = (P_{ij}) \neq \emptyset$. Namely, move the cycle α_j along the fibres until it lies on $\varphi^{-1}(P_{ij}) \subset M_i$, and there call it β_{ij}. By my construction of the patching of M_i and M_j, we know that β_{ij} is what I called η, while α_i is ξ. Now compute the subgroup K_i of $H_1(M_i)$ defined by

$$\begin{array}{ccccccccc} 0 & \to & K_i & \longrightarrow & H_1(M_i) & \to & H_1(E_i) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K & \longrightarrow & H_1(M) & \to & H_1(\cup E_i) & \to & 0. \end{array}$$

As above, let U_{ij} be a small disc on E_i about P_{ij}, and $E_i^* = E_i - \cup U_{ij}$, and $M_i^* = \varphi^{-1}(E_i^*)$. Then M_i^* is a deformation retract of M_i, and is, on the one hand canonically the restriction of the bundle S_i to E_i^*, and on the other hand uncanonically homeomorphic to $S^1 \times E_i^*$. In this last description, α_i is canonically identified to $S^1 \times (\text{point})$, while β_{ij} are identified to $(\text{point}) \times \partial(U_{ij})$ only up to adding a multiple of α_i. Therefore we see that K_i is generated by α_i, β_{ij}, with one relation [1]

$$\sum_j \beta_{ij} + N\alpha_i = 0, \text{ some } N.$$

To evaluate N, note that β_{ij} considered as cycles in S_i are locally contractible (i.e. in the neighborhood of $\varphi^{-1}(P_{ij})$ described by my plumbing fixture). It is well known that when the oriented fundamental 2-cycle of E_i is lifted to S_i, its boundary is $(E_i^2)\alpha_i$. Therefore, this same lifting in M_i^* will have boundary $\sum_j \beta_{ij} + (E_i^2)\alpha_i$. Now by the Mayer-Vietoris sequence, $H_1(M)$ is generated by $H_1(M_i)$, hence K is by K_i, and has extra relations imposed by the identification of cycles on $M_i \cap M_j$. Since $H_1(M_i \cap M_j)$ is generated by β_{ij} and β_{ji}, these relations are implicit in our choice of generators.

As a consequence of our result, since $\det(E_i . E_j) = \mu \neq 0$, K is a finite group of order μ, and is the torsion subgroup of $H_1(M)$.

Now consider the case E_i rational, and $\cup E_i$ tree-like. We shall compute $\pi_1(M)$, using $\pi_1(M_i)$ as building blocks. In order to keep these various groups, with their respective base points, under control, it is necessary to define a skeleton of basic paths leading throughout E_i. Let $Q_i \in E_i - \bigcup_{j\neq i} E_j$ be chosen as base point in E_i. On E_i, choose a path l_i as illustrated in Diagram II touching on each $P_{ij} \in E_i$. Lift all the l_i together into M by a map s, so that $\varphi(s(l_i)) = l_i$, and so that at $\varphi^{-1}(P_{ij})$, $s(l_i) \cap s(l_j) \neq \emptyset$. Choose, e.g. $s(Q_1)$ as base point for all of M. Let $G = \cup l_i$. Now the lifting s enables us to give the following recipe for paths α_i:

1. Go along $s(G)$ from $s(Q_1)$ to a point P in M_i.
2. Go once around the fibre of M_i through P in the canonical direction explained above.
3. Go back to $s(Q_1)$ along $s(G)$.

(1) In the map $H_1(E_i^*) \to H_1(E_i)$, the kernel is generated by $\{\partial(U_{ij})\}$ with the single relation $\Sigma_{j\neq i}\partial(U_{ij}) = \partial(\text{fundamental 2-cycle of } E_i^*) \sim 0$.

This is clearly independent of the choice of P.

Our result can now be stated: firstly, the α_i generate π_1; secondly, their only relations are (*a*) α_i and α_j commute if $E_i \cap E_j \neq \emptyset$, (*b*) if $k_i = (E_i^2)$, and $E_{j_1}, E_{j_2}, \ldots, E_{j_m}$ are those E_j intersecting E_i, written in the order in which they intersect l_i, then

$$e = \alpha_{j_1} \alpha_{j_2}, \ldots, \alpha_{j_m} \alpha_i^{k_i}.$$

To prove this, we use the following theorem of Van Kampen (see [8], p. 30): if X and Y are subcomplexes of a complex Z, and $Z = X \cup Y$, while $X \cap Y$ is connected, then $\pi_1(Z)$ is the free product of $\pi_1(X)$ and $\pi_1(Y)$ modulo amalgamation of the sub-

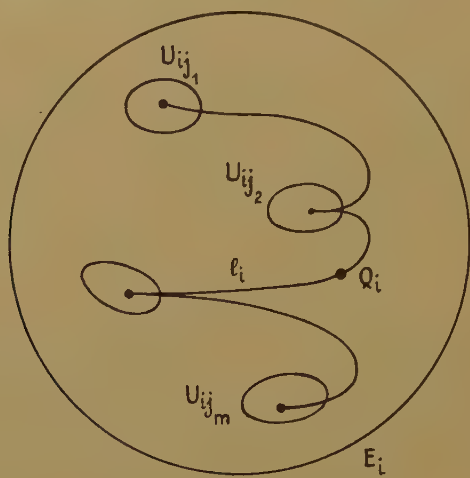

Fig. 2

groups $\pi_1(X \cap Y)$. Now since E_i is tree-like, M can be gotten from the M_i by successively joining on a new M_i with *connected* intersection with the part so far built up. Let $\pi_1(M_i)$ be mapped into $\pi_1(M)$ by mapping a loop in M_i with base point $s(Q_i)$ to one in M with base point $s(Q_1)$ by simply tagging on to both ends of it the section of $s(G)$ joining these two points. Then $\pi_1(M)$ is simply the free product of the $\pi_1(M_i)$ with amalgamation of the loops in $M_i \cap M_j$. Now recalling the structure of M_i^*, we have an exact sequence that splits:

$$0 \to \pi_1(S^1) \to \pi_1(M_i^*) \xrightarrow{\pi} \pi_1(E_i^*) \to 0$$

(S^1 the fibre of M_i, a 1-sphere). The path α_i is clearly a generator of $\pi_1(S^1)$ here, and hence in the center of $\pi_1(M_i^*)$.

Now the important thing to notice is that if E_i meets E_j, then α_j in $\pi_1(M_j)$ can be moved by modifying the point P on $s(G)$ where α_j detours around the fibre S^1; in particular, it may do this at $s(l_i) \cap s(l_j)$. In that position the loop α_j may be regarded canonically as in $\pi_1(M_i)$. Under the identification of $\pi_1(M_i)$ to $\pi_1(M_i^*)$ and the projection π of this group onto $\pi_1(E_i^*)$, what happens to the loop α_j? Recalling the patching map on the boundaries of M_i^* and M_j^* which was examined above, we see that this path proceeds along G from Q_i to near P_{ij}, then circles around the boundary of U_{ij}

in a positively oriented direction, then returns along G to Q_i. Referring again to our diagram, we see the relation $e=\pi(\alpha_{j_1}).\pi(\alpha_{j_2}).\ldots.\pi(\alpha_{j_m})$. Now it is well-known that these loops $\pi(\alpha_{j_k})$ generate the fundamental group of the m-times punctured sphere, and that this is the unique relation. Consequently, looking at the above exact sequence, it is clear that $\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_m}$ (when distorted into M_i as indicated above) generate $\pi_1(M_i)$. Moreover, the only relations among these generators are, therefore, that α_i and α_{j_k} commute, and $\alpha_{j_1}\ldots\alpha_{j_m}\in\pi_1(S^1)$, i.e. $=\alpha_i^N$. But, using our results on $H_1(M)$, $N=-(E_i^2)$.

It follows that α_i generate $\pi_1(M)$ with relations (a) and (b), and that the only additional relations are those coming from the amalgamation of $\pi_1(M_i\cap M_j)=\mathbf{Z}+\mathbf{Z}$. But α_i and α_j are generators here, and as loops in M_i and M_j, these have already been identified. Hence we are through, Q.E.D.

II. — ALGEBRO-GEOMETRIC SIGNIFICANCE OF $H_1(M)$

(a) Local Analytic Picard Varieties and Unique Factorization.

We shall study in this section two questions of algebro-geometric interest in the solution of which the topological structure of M, in particular its homological structure, is reflected. The first of these is the problem of the local Picard Variety at $P\in F$. Generally speaking, this, as a group, should be the group of *local* divisors at P modulo local linear equivalence to zero. (We shall be more precise below.) However, if by divisor one refers to an algebraic divisor and by local one means in the sense of the Zariski topology, one sees by example that the resulting group has little significance: it is not local enough. Ideally, one should mean by an irreducible local divisor a minimal prime ideal in the formal completion of the local ring of the point in question. However, I have been unable to establish the structure of the resulting Picard group. A compromise between these two groups is possible over the complex numbers. Take as divisors *analytic* divisors, and the usual complex topology to interpret local. There results a local *analytic* Picard variety that is quite accessible. In this section, we shall first analyze the group of local *analytic* divisors near $\cup E_i$ modulo local linear equivalence and then consider the singular point P. Here by local analytic divisors we mean formal sums of irreducible analytic divisors defined in a neighborhood of $\cup E_i$ (including the divisors E_i themselves). Such a sum, $\Sigma n_i D_i$, is said to be locally linearly equivalent to zero if there exists a neighborhood U of $\cup E_i$ where all D_i are defined and a meromorphic function f on U such that $(f)=\Sigma n_i(D_i\cap U)$. This quotient we shall call the local analytic Picard Variety at $\cup E_i$, or Pic $(\cup E_i)$.

Denote by Ω the sheaf of germs of holomorphic functions on F'; by $\Omega^*\subset\Omega$ the sheaf of germs of non-zero holomorphic functions. One has the usual exact sequence:

$$0\to\mathbf{Z}\to\Omega\xrightarrow{\exp(2\pi ix)}\Omega^*\to 0$$

where $\mathbf{Z}$ is the constant sheaf of integers. Let $\pi\colon F'\to F$ be the regular projection from the non-singular surface F' to the singular F.

Proposition. — $\mathrm{Pic}(\cup E_i) \simeq (R^1\pi)(\Omega^*)_P$.

Proof. — Define $\mathrm{Pic}(\cup E_i) \to (R^1\pi)(\Omega^*)_P$, by associating to $\Sigma n_i D_i$, defined in $U \supset \cup E_i$, the following 1-cocycle: assume $P \in V$, $\pi^{-1}(V) \subset U$, assume f_j is a local equation for $\Sigma n_i D_i$ in V_j, $\{V_j\}$ a covering of V, then $\{f_{j_1}/f_{j_2}\} \in H^1(\{V_j\}, \Omega^*)$ induces an $\alpha \in H^1(\pi^{-1}(V), \Omega^*)$, hence an $\alpha' \in (R^1\pi)(\Omega^*)_V$, hence an $\alpha'' \in (R^1\pi)(\Omega^*)_P$. It is well known that $\alpha \in H^1(\pi^{-1}(V), \Omega^*)$ is uniquely determined by $\Sigma n_i D_i$, hence so is α''.

To see that $\Sigma n_i D_i \to \alpha''$ is $1-1$, say $\alpha'' = 0$. Therefore $\exists V' \subset V$ say, $\mathrm{Res}_{V'} \alpha' = 0$, i.e. $\mathrm{Res}_{\pi^{-1}(V')}(\alpha) = 0$. Therefore the covering $\{V_j \cap \pi^{-1}(V')\} = \{V'_j\}$ has a refinement $\{V''_k\}$ such that there exist non-zero functions g_k on V''_k such that $g_{k_1}/g_{k_2} = f_{\tau k_1}/f_{\tau k_2}$ (for some map τ from the indices of $\{V''_k\}$ to those of $\{V_j\}$ such that $V''_k \subset V'_{\tau k}$). Therefore $f = \dfrac{f_{\tau k}}{g_k}$ defines a function throughout $\pi^{-1}(V')$ such that $(f) = \Sigma n_i D_i$.

To see that $\Sigma n_i D_i \to \alpha''$ is onto $(R^1\pi)(\Omega^*)_P$, let $\beta'' \in (R^1\pi)(\Omega^*)_P$ be represented by $\beta \in H^1(\pi^{-1}(V), \Omega^*)$ and let this define the line bundle L over $\pi^{-1}(V)$ in the usual way. Let $\mathscr{I}$ be the sheaf of germs of cross-sections of L: a coherent sheaf. Now by a result of Grauert and Remmert (cf. Borel-Serre [2], p. 104), $(R^0\pi)(\mathscr{I})$ is coherent on F. But $(R^0\pi)(\mathscr{I})$ is not the zero sheaf on F (at all points $Q \neq P$, $\mathscr{I}_Q \simeq (R^0\pi)(\mathscr{I})_Q$), hence there exists some element $S \in (R^0\pi)(\mathscr{I})_P$, $S \neq 0$. S corresponds to a section in $\mathscr{I}_{\pi^{-1}(V')}$, for some open $V' \ni P$, $V' \subset V$. Therefore, the line bundle $L \mid \pi^{-1}(V)$ has a section S. But if β is represented by a cocycle f_{ij} with respect to a covering $\{V_i\}$ of V, then S is given by a set of holomorphic functions f_i on V_i such that $f_j = f_i(f_{ij})$. It follows that $f_i = 0$ define a divisor which is represented by β.

A. Grothendieck has posed the problem, for any proper map $f: V_1 \to V_2$ (onto), to define a relative Picard Variety of the map f. It seems clear, in the classical case, that if Ω^* is the sheaf of holomorphic units on V_1, $(R^1 f)(\Omega^*)$ is the logical choice although no nice properties have been established in general so far as the writer knows. In our case, $(R^1 f)(\Omega^*)_Q$, for $Q \neq P$, is simply (1), but at P, we have seen it to be $\mathrm{Pic}(\cup E_i)$. We now wish to show that in our case, $(R^1 f)(\Omega^*)_P$ is an analytic group variety. This is seen by the exact sequence for derived functors:

$$\begin{aligned} 0 &\to (R^0\pi)(\mathbf{Z}) \to (R^0\pi)(\Omega) \xrightarrow{\varphi} (R^0\pi)(\Omega^*) \to \\ &\to (R^1\pi)(\mathbf{Z}) \xrightarrow{\chi} (R^1\pi)(\Omega) \to (R^1\pi)(\Omega^*) \xrightarrow{\psi} \\ &\to (R^2\pi)(\mathbf{Z}) \to \dots \end{aligned}$$

(i) Note first that if $x \in (R^0\pi)(\Omega^*)_P$, then x is a non-zero function on $\pi^{-1}(V)$, $P \in V$, and necessarily constant on $\cup E_i$ which is connected and compact, therefore, at least on some $\pi^{-1}(V')$, $P \in V' \subset V$, $x = \exp(2\pi i y)$, y a holomorphic function on $\pi^{-1}(V')$, hence $x = \varphi(y)$, $y \in (R^0\pi)(\Omega)_P$.

(ii) Note secondly that $(R^i\pi)(\mathbf{Z})_p \simeq H^i(\cup E_i, \mathbf{Z})$, since for $P \in V$, V small, $\pi^{-1}(V)$ is contractible to $\cup E_i$.

(iii) Note thirdly that if $i>0$, $(R^i\pi)(\Omega)_Q=(0)$ for $Q\neq P$, and being a coherent sheaf, for $Q=P$ must be a finite dimensional vector space over $\mathbf{C}$.

(iv) Note fourthly that if $\gamma\in H^2(\cup E_i, \mathbf{Z})\simeq(R^2\pi)(\mathbf{Z})_P$, there exists $\alpha\in(R^1\pi)(\Omega^*)_P$ such that $\psi\alpha=\gamma$. To show this, note that $H^2(\cup E_i, \mathbf{Z})\simeq\mathbf{Z}^n$, ($n=$ number of irreducible curves in $\cup E_i$) with generators γ_i whose value on the 2-cycle E_j is δ_{ij}; it is enough to verify it for the generators γ_i. But let D_i be an irreducible analytic curve through $Q\in E_i-\bigcup_{j\neq i} E_j$, with a simple point at Q, and tangent transversal to that of E_i at Q. If $D_i\to\alpha_i\in(R^1\pi)(\Omega^*)$, I claim $\psi\alpha_i=\gamma_i$. This is left to the reader. Therefore, we obtain

$$0\to H^1(\cup E_i, \mathbf{Z})\xrightarrow{\chi}(R^1\pi)(\Omega)_P\to \mathrm{Pic}\,(\cup E_i)\to H^2(\cup E_i, \mathbf{Z})\to 0$$
$$(R^1\pi)(\Omega)_P \simeq \mathbf{C}^N, \text{ some } N.$$

(v) Note lastly that χ maps $H^1(\cup E_i, \mathbf{Z})$ into a *closed* subgroup of $(R^1\pi)(\Omega)_P$, hence the connected component of $\mathrm{Pic}(\cup E_i)$ is an analytic group. If this were false, there would be a *real* sum of elements of $H^1(\cup E_i, \mathbf{Z})$ that was zero without having to be, i.e. $\{\alpha_{ij}\}\in H^1(\pi^{-1}(V), \mathbf{R})$ (with respect to some covering $\{U_i\}$) such that $\{\alpha_{ij}\}\sim 0$ in the sheaf Ω (in some $\pi^{-1}(V')$, $V'\subset V$). In other words, $\alpha_{ij}=f_i-f_j$, f_i holomorphic in U_i. But let p_i be a real, C^∞ function on U_i such that $\alpha_{ij}=p_i-p_j$ (Poincaré's lemma). Then $f_i-p_i=F$, $df_i=\omega$ and $dp_i=\eta$, are defined all over $\pi^{-1}(V')$, $\omega-\eta=dF$. I claim actually all the periods of η are zero (which implies $\eta=df$, and $\{\alpha_{ij}\}\sim 0$ in $H^1(\cup E_i, \mathbf{R})$ and we are through). First of all, the periods of η equal those of ω. Look at its periods on the 1-cycles of any E_i: since η is real, all the periods of the holomorphic differential ω are also real. But it is wellknown that then all the periods of ω must be identically zero, and therefore ω reduces to *zero* on paths in E_i. Since this is true for all i, ω has no periods along *any* path in $\cup E_i$, and since $\pi^{-1}(V')$ is contractible to $\cup E_i$, ω has no periods at all. Therefore neither does η and we are through.

There is another way of looking at $\mathrm{Pic}(\cup E_i)$. Namely, let $\mathfrak{o}$ be the local ring of (convergent) holomorphic functions at P, i.e. $(R^0\pi)(\Omega)_P$ (by the theorem of Riemann, cf. the report of Behnke and Grauert ([1], p. 18)). Now every divisor D' in $\pi^{-1}(V')$, except for the E_i's, defines a divisor D in V', hence a minimal prime ideal $\mathfrak{p}$ in $\mathfrak{o}$. Let us set Pic(P) equal to the group of ideal classes in $\mathfrak{o}$: i.e. to the semi-group of pure rank 1 ideals $\mathfrak{a}$ of $\mathfrak{o}$, modulo the principal ideals [1]. Then the association of D to $\mathfrak{p}$ defines a map from $\mathrm{Pic}(\cup E_i)\to\mathrm{Pic}(P)$, (if we define the image of each E_i to be (1), the identity). This is quite clear once one sees that every meromorphic function f in $\pi^{-1}(V)$ is a quotient

[1] The composition law is the "Kronecker" product treated so elegantly by Hermann Weyl [12], cf. chapter 2, namely:

$$(\mathfrak{a}, \mathfrak{b})\to \text{rank 1 component of } \mathfrak{a}.\mathfrak{b}$$
$$= \bigcup_{n=1}^{\infty} (\mathfrak{a}.\mathfrak{b}) : \mathfrak{m}^n$$

where $\mathfrak{m}$ = maximal ideal of $\mathfrak{o}$
(:) = residual quotient operation.

of two holomorphic functions in some $\pi^{-1}(V')$, $V' \subset V$: but given f, consider the coherent sheaf $\mathscr{J}$ given by $\{g \mid (fg)$ is a positive divisor$\}$. $(R^0\pi)(\mathscr{J})$ is coherent, hence there exists $g_1 \in (R^0\pi)(\mathscr{J})_P$, and if $fg_1 = g_2$, then $f = g_2/g_1$ is the desired decomposition. Now the map $\mathrm{Pic}(\cup E_i) \to \mathrm{Pic}(P)$ is onto as every minimal prime ideal $\mathfrak{p} \subset \mathfrak{o}$ defines some divisor through P. Its kernel is immediately seen to be generated by the E_i themselves. Hence we see

Proposition:
$$\frac{\mathrm{Pic}(\cup E_i)}{\{\Sigma n_i E_i\}} \simeq \mathrm{Pic}(P)$$

Corollary. — We have
$$0 \to H^1(\cup E_i, \mathbf{Z}) \to (R^1\pi)(\Omega)_P \xrightarrow{\varphi} \mathrm{Pic}(P) \xrightarrow{\psi} H_1(M)_0 \to 0$$

where $H_1(M)_0 =$ torsion subgroup of $H_1(M)$ and ψ associates to the divisor D through P, the 1-cycle $D \cap M$.

Proof of Corollary: Note that $\Sigma n_i E_i$ is never in the image of $(R^1\pi)(\Omega)_P$ since that would require $(\Sigma n_i E_i, E_j) = 0$ for all j. To see the exactness at ψ, note that the co-kernel of φ is obtained by associating to a divisor $\Sigma n_i D_i$ (where we may assume $E_i \cap E_j \cap (\bigcup_l \mathrm{Supp}\, D_l) = \emptyset$, all $i \neq j$) the formal sum
$$\sum_k \Big(\sum_i n_i D_i \,.\, E_k\Big)\gamma_k \text{ modulo } \Big\{\sum_k (E_i \,.\, E_k)\gamma_k\Big\},$$

the γ_k as in (iv) above. But ψ is given by associating to $\Sigma n_i D_i$, the element
$$\sum_k (\Sigma n_i D_i \,.\, E_k)\alpha_k,$$

in terms of our basis for $H_1(M)_0$ in (I); but by our enumeration of the relations on the α_k we see γ_k can be interchanged with α_k.

Do these results have purely algebraic counterparts? First, note that it is hopeless to expect that the ideal structure of $\mathfrak{o}_0$ (= algebraic local ring of P on F) will reflect the homology of the singularity so well. This is seen in the following example: Take a non-singular cubic curve E in the projective plane, and let $P_1, \ldots, P_{15}$ be points on E in general position except that on E the divisor $\Sigma_1^{15} P_i \equiv 5 \times$ (plane section). Blow up every point P_i to a divisor E_i, and call F′ the resulting surface. On F′, the proper transform E′ of E is exceptional: it is shrunk by the linear system of quintics through the P_i. Then $E_i - E_j$ as a divisor in Pic(E′) is in the component of the identity, but as an algebraic divisor is not algebraically locally equivalent to zero: in fact F′ is regular, hence algebraic and linear equivalence are the same, but since $\mathrm{Tr}_{E'}(E_i - E_j) \not\equiv 0$, $E_i - E_j$ is not locally linearly equivalent to zero.

However, I conjecture that the ideal class group of $\mathfrak{o}^*$ (= completion of $\mathfrak{o}_0$ and $\mathfrak{o}$) is *identical* to that of $\mathfrak{o}$, and that sums of formal branches through $\cup E_i$ modulo holomorphic linear equivalence (in the sense of Zariski [17]) gives $\mathrm{Pic}(\cup E_i)$. If this is so, it should give $\mathrm{Pic}(\cup E_i)$ an *algebraic* structure, which would be a decided improvement on our results. At present, I am unable to prove these statements.

(*b*) **Intersection Theory on Normal Surfaces.**

We consider here the problem of defining, for divisors A, B through P on F, (*a*) total transforms A′, B′ on F′, and (*b*) intersection multiplicities $i(A.B; P)$. This problem has been posed by Samuel (see [7]) and considered by J. E. Reeve [19]. In this case, I suggest the following as a canonical solution:

a) To define $A' = A'_0 + \Sigma r_i E_i$, where A'_0 is the proper transform of A, require

$$(A'.E_i) = 0, \; i = 1, 2, \ldots, n,$$

or

$$(A'_0.E_i) + \sum_j r_j(E_j.E_i) = 0, \; i = 1, 2, \ldots, n.$$

Since $\det(E_i.E_j) = \mu \neq 0$, this has a unique solution.

b) To define $i(A.B; P)$, set it equal to

$$\begin{aligned}&(A'.B') \text{ over } P\\ &= \sum_{P' \text{ over } P} [i(A'_0.B'_0; P') + \Sigma r_i i(E_i.B'_0; P')]\\ &= \sum_{P' \text{ over } P} [i(A'_0.B'_0; P') + \Sigma s_i i(A'_0.E_i; P')]\end{aligned}$$

where

$$A' = A'_0 + \Sigma r_i E_i; \quad B' = B'_0 + \Sigma s_i E_i.$$

We note the following properties:

(i) $A = (f)_F$, then $A' = (f)_{F'}$; hence $A \equiv B$ implies $A' \equiv B'$.

Proof. — For $((f)_{F'}.E_i) = 0$.

(ii) A effective, then all r_i are positive.

Proof. — Say some $r_i \leq 0$. Say also $r_i/m_i \leq r_j/m_j$, all j, where the m_j are the same as in the proof of negative definiteness. Then we see:

$$\begin{aligned}0 &\geq \sum_j r_j (E_j.E_i) = \sum_j r_j/m_j \, (m_j E_j.E_i),\\ &\geq r_i/m_i \sum_j (m_j E_j.E_i) \geq 0.\end{aligned}$$

Therefore, if $E_i \cap E_j \neq \emptyset$, $r_i/m_i = r_j/m_j$ and $r_j \leq 0$. As $\cup E_i$ is connected, this gives ultimately $r_i/m_i = R$, independent of i. But then also $(\Sigma m_j E_j.E_i) = 0$, all i, which contradicts property (*c*) in the proof just referred to.

(iii) $i(A.B; P)$ is symmetric and distributive.

(iv) A and B effective, then $i(A.B; P)$ is greater than 0.

(v) $i(A.B; P)$ independent of the choice of F′.

Proof. — To show this, it suffices, since any two non-singular models are dominated by a third, see Zariski [15], to compare F′ with F″ gotten by blowing up some point P′ over P. But let A′, B′ be the total transforms of A, B on F′, and A″, B″ those on F″, and let T be the map from F″ to F′. Then with respect to T, A″ is the total transform

of A' on F'', and B'' that of B'. In that case it is well-known that, for any point set S in F' (including all the points of any common components of A', B'), $(A'.B')_S = (A''.B'')_{T^{-1}(S)}$.

(vi) A' is integral if and only if $\Sigma(A'_0.E_i)\alpha_i = 0$ in $H_1(M)$.

Proof. — $\Sigma(A'_0.E_i)\alpha_i = 0$ if and only if there are integers k_j such that

$$(A'_0.E_i) = \Sigma k_j(E_j.E_i),$$

i.e. if the relation $\Sigma(A'_0.E_i)\alpha_i = 0$ is an integral sum of the relations defining $H_1(M)$. But this is equivalent to $(A'_0 + \Sigma k_j E_j.E_i) = 0$ for all i, i.e. $A' = A'_0 + \Sigma k_j E_j$, k_j integral. Q.E.D.

The element $\Sigma(A'_0.E_i)\alpha_i$ has this simple interpretation: if M is chosen near enough to P, it represents the 1-cycle $A \cap M$. We see that this is again the fundamental map: (Group of Local Divisors at $P) \to H_1(M)$ considered in the final corollary of part (*a*). By the results of part (*a*), moreover, we can interpret (vi) as saying: A' is integral if and only if A is locally analytically equivalent to zero (i.e. A is in the connected component of Pic(P)). Essentially, our definition of intersection multiplicity on a normal surface is the unique linear theory that has the correct limiting properties for divisors that can be analytically deformed off the singular points.

III. — THE CASE $\pi_1(M) = (e)$

We shall prove the following theorem, stronger than that announced above:

Theorem. — Let F be a non-singular surface, and E_i, $i = 1, 2, \ldots, n$, a connected collection of non-singular curves on F, such that $E_i \cap E_j$ is empty, or consists of one point on a transversal intersection, and $E_i \cap E_j \cap E_k$ is always empty. Let M be a tubular neighborhood of $\cup E_i$, as defined in section I. If (*a*) $\pi_1(M) = (e)$, and (*b*) $((E_i.E_j))$ is negative definite, then $\cup E_i$ is exceptional of first kind, i.e. is the total transform of some simple point on a surface dominated by F and birational to it.

Proof. — As above, $\pi_1(M) = (e)$ implies that all E_i are rational, and connected together as a tree. Now suppose that $\cup E_i$ is not exceptional of first kind. Assume that among all collections of E_i with all the properties of the theorem, there is no collection not exceptional with *fewer* curves E_i. As a consequence, no E_i of our collection has the two properties (*a*) $(E_i^2) = -1$, (*b*) E_i intersects at most two other E_j. For if it did, one could shrink E_i by Castelnuovo's criterion, preserving all the properties required (that the negative definiteness is preserved is clear as follows: the self-intersection of a cycle of the E_j's on the blown down surface equals the self-intersection of its total transform on F which must be negative). We allow the case where there is only one E_i. Now the central fact on which this proof is based is the following group-theoretic proposition:

Proposition. — Let G_i, $i = 1, 2, 3$, be non-trivial groups, and a_i an element of G_i. Then denoting the free product of A and B by $A * B$, it follows $G_1 * G_2 * G_3/\text{modulo } (a_1 a_2 a_3 = e)$ is non-trivial.

Proof. — First of all, if $\infty \geq n_1, n_2, n_3 > 1$, then $Z_{n_1} * Z_{n_2} * Z_{n_3}/(a_1 a_2 a_3 = e)$ is non-trivial, where Z_k denotes the integers modulo k, and each a_i is a generator. For, as a matter of fact, these are well-known groups easily constructed as follows: choose a triangle with angles π/n_1, π/n_2, and π/n_3 (modular if some $n_i = \infty$), in one of the three standard planes. Reflections in the three sides of the triangle generate a group of motions of the plane, and the group we seek is the subgroup, of index 2, of the orientation preserving motions in this group. Secondly, reduce the general statement to this case by means of:

($\#$) If n = order of a_1 in G_1, and a_1 is identified to a generator of $Z_n \subseteq G_1$, then $G_1 * G_2 * G_3/(a_1 a_2 a_3 = e)$ trivial $\Rightarrow Z_n * G_2 * G_3/(a_1 a_2 a_3 = e)$ trivial.

To show this, let $H = G_2 * G_3/((a_2 a_3)^n = e)$, and note that H is isomorphic to $Z_n * G_2 * G_3/(a_1 a_2 a_3 = e)$. Let n' be the order of a_1 in H. Then $G_1 * G_2 * G_3/(a_1 a_2 a_3 = e)$ is the free product of $G_1/(a_1^{n'} = e)$ and H with amalgamation of the subgroups generated by $a_2 a_3$ and a_1^{-1}. But by O. Schreier's construction of amalgamated free products (see [5], p. 29) this is trivial only if H is, hence ($\#$). Now the proposition is trivial if any $a_i = e$; hence let n_i = order $(a_i) > 1$. By ($\#$) iterated, $G_1 * G_2 * G_3/(a_1 a_2 a_3 = e)$ trivial implies $Z_{n_1} * Z_{n_2} * Z_{n_3}/(a_1 a_2 a_3 = e)$ trivial, which is absurd. Q.E.D.

Returning to the theorem, we wish to show the absurdity of $\pi_1(M) = (e)$, while no E_i is such that (*a*) $(E_i^2) = -1$, and (*b*) E_i meets at most two other E_j. There are two cases to consider: either *some* E_i meets three or more other E_j; or every E_i meets at most two other E_j (this includes the case of only one E_i).

Case 1. — Let E_1 meet $E_2, \ldots, E_m$, where m is at least 4. For $i = 2, 3, \ldots, m$, let T_i be the set of E_j's (besides E_1) such that E_j is connected to E_i by a series of E_k other than E_1. The T_i's are disjoint. Let M_i be the manifold bounding a neighborhood of T_i as above. Let $G_i = \pi_1(M_i)$, and $G = \pi_1(M)$/modulo $\alpha_1 = e$, where α_1 represents, as in (I), the loop about E_1. Then by the results of (I),

$$G = G_2 * G_3, \ldots, * G_m/(\alpha_2 \alpha_3 \ldots \ldots \alpha_m = e),$$

if the G_i are ordered suitably, and α_i in G_i represents a loop about E_i. Now $m \geq 4$, and $\pi_1(M) = (e)$, hence $G = (e)$, hence by the above theorem, there exists an i (say $i = 2$) such that $G_2 = \pi_1(M_2) = (e)$. By the induction assumption, the tree of curves T_2 is exceptional of first kind. Therefore, by Zariski's theorem on the factorization of anti-regular transformations on non-singular surfaces (see [18]), some E_j in T_2 enjoys the properties (*a*) and (*b*) with respect to T_2. Then E_j would also enjoy them in $\cup E_i$ (which is impossible) *unless* $E_j = E_2$, in which case E_j could meet only two other E_k (say E_{m+1}, E_{m+2}) in T_2, but would meet *three* other E_k in $\cup E_i$. Pursuing this further, apply the same reasoning to the curve E_2 which meets exactly three other E_k. Again, either some curve shrinks, or else either E_1, E_{m+1}, or E_{m+2} has in any case property (*a*), i.e. self-intersection -1. But then compute $((E_2 + E_i)^2)$ ($i = 1, m+1$, or $m+2$ according as which E_i has property (*a*)), and we get 0, contradicting negative definiteness of the intersection matrix.

Case 2. — It remains to consider the case where no E_i intersects more than two others. Then the E_i are arranged as follows:

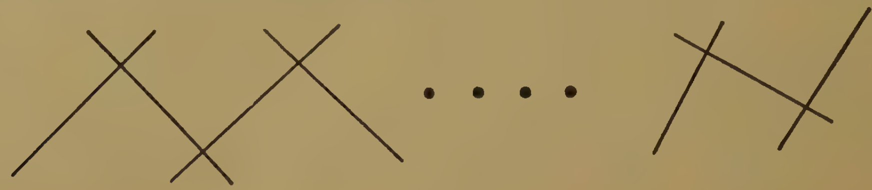

Fig. 3

In this case, it is immediate that π_1 is commutative, hence $=H_1$. It is given (in additive notation) by the equations:

$$\begin{array}{ll}
k_1\alpha_1 - \alpha_2 \ldots\ldots\ldots\ldots & =0 \\
-\alpha_1 + k_2\alpha_2 - \alpha_3 \ldots\ldots & =0 \\
\quad - \alpha_2 + k_3\alpha_3 \ldots\ldots & =0 \\
\ldots\ldots\ldots\ldots\ldots\ldots & \\
\quad -\alpha_{n-1} + k_n\alpha_n & =0,
\end{array}$$

where $k_i = -(E_i^2)$. Assume all $k_i \geq 2$, and prove

$$\mu = \det\begin{pmatrix} k_1 & -1 & 0 & 0 \ldots\ldots & 0 \\ -1 & k_2 & -1 & 0 \ldots\ldots & 0 \\ 0 & -1 & k_3 & -1 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 \ldots & \ldots & 0 & -1 & k_n \end{pmatrix} > 1,$$

hence the equations have a solution mod μ. To show this, use induction on n, using the stronger induction hypothesis $k_1 > 1$, $k_2, \ldots, k_n \geq 2$, allowing k_i to be rational. Then note the identity:

$$\det\begin{pmatrix} k_1 & -1 & 0 \ldots 0 \\ -1 & k_2 & -1 \ldots 0 \\ \ldots & \ldots & \ldots \\ 0 \ldots\ldots & -1 & k_n \end{pmatrix} = k_1 \det\begin{pmatrix} (k_2 - 1/k_1) & -1 \ldots 0 \\ -1 & k_3 \ldots 0 \\ \ldots & \ldots \\ 0 \ldots\ldots -1 & k_n \end{pmatrix}$$

This completes the proof of our theorem.

Corollary. — P a normal point of an algebraic surface F. If F has a neighborhood U homeomorphic to a 4-cell, P is a simple point of F.

Proof. — Let W be the intersection of an affine ball about P with F, as considered in the introduction, and so small that its boundary M lifted to a non-singular model F′ dominating F qualifies as a tubular neighborhood of the total transform of P. It suffices to show that $\pi_1(M) = (e)$, in view of the theorem just proven. Let U′ be a 4-cell-neighborhood of P contained in W, and let W′ be an affine ball about P contained in U′. We have constructed in section I a continuous map ψ from U′ — (P) to M that

induces the canonical identification of M as the boundary of W′ to M (as the boundary of W). Therefore if γ is any path in M, regard γ as a path in the boundary of W′; as a path in $U' - (P)$ (which is homotopic to a 3-sphere) it can be contracted to a point; but then ψ maps this homotopy to contraction of γ as a path in M. Q.E.D.

IV. — AN EXAMPLE

It is instructive to note that there exist singular points P, for which $H_1(M) = (0)$, while, of course, $\pi_1(M) \neq (e)$. Take P to be the origin of the equation $0 = x^p + y^q + z^n$, where p, q, and n are pairwise relatively prime. Look at the equation as $-(z)^n = x^p + y^q$; this shows that M is an n-fold cyclic covering of the 3-sphere $|x|^2 + |y|^2 = 1$, x, y complex, branched along the points $x^p + y^q = 0$, i.e. along a torus knot, K, in S^3. Therefore M is a manifold of the type considered by M. Seifert [20], p. 222; he shows $H_1(M) = (0)$.

The singular point $0 = x^2 + y^3 + z^5$ is of particular interest as illustrating the possibility of a singular point on a surface whose local analytic Picard Variety is trivial contrary to a conjecture of Auslander. To show Pic(P) $(P = (0, 0, 0))$, is trivial amounts to showing $(R^1\pi)(\Omega)_P = (0)$, where $\pi\colon F' \to F$ is the map from a non-singular model to $0 = x^2 + y^3 + z^5$ (since we know $H_1(M) = (0)$ already). Let us choose a slightly better global surface F (our statement being local, we are free to choose a different model of $k(F)$ outside a neighborhood of P): namely take F_0 to be the double plane with sextic branch locus $B : u(u^2y^3 + z^5)$, where u, y, z are homogeneous coordinates. F_0 has two singularities: one is over $y = z = 0$ and this is P; the other is over $u = z = 0$ — call it Q. Let F_1 be the result of resolving Q alone, and F_2 be the non-singular surface obtained by resolving P and Q. Let $\pi : F_2 \to F_1$. We must show $(R^1\pi)(\Omega_{F_2})_P \simeq (0)$. But since $(R^1\pi)(\Omega_{F_2})$ is (0) outside of P, it is equivalent to show $H^0(F_1, (R^1\pi)(\Omega_{F_2})) = (0)$. First of all, note that F_2 is birational to P^2: indeed $0 = x^2 + y^3 + z^5$ is uniformized by the substitution:

$$x = 1/u^3v^5(u+v)^7,\ y = -1/u^2v^3(u+v)^5,\ z = -1/uv^2(u+v)^3.$$

Therefore $0 = H^1(F_2, \Omega_{F_2}) = H^2(F_2, \Omega_{F_2})$. Now consider the Spectral Sequence of Composite Functors:

$$H^i(F_1, (R^j\pi)(\Omega_{F_2})) \underset{i}{\Rightarrow} H^k(F_2, \Omega_{F_2}).$$

Noting that $(R^0\pi)(\Omega_{F_2}) = \Omega_{F_1}$, it follows:

a) $H^1(F_1, \Omega_{F_1}) = (0)$
b) $d_2^{0,1}\colon H^0(F_1, (R^1\pi)(\Omega_{F_2})) \to H^2(F_1, \Omega_{F_1})$
is $1-1$, onto.

Therefore, it suffices to show $H^2(F_1, \Omega_{F_1}) = (0)$, or $0 \geqslant p_a(F_1)$ $(= \dim H^2 - \dim H^1)$. Now unfortunately $p_a(F_0) = 1$, since, in general, if G is a double plane with branch locus of order $2m$, $p_a(G) = (m-1)(m-2)/2$ (none of the singularities of G being resolved,

of course) (1). To compute $p_a(\mathrm{F}_1)$, embed F_0 in a family of double planes $\mathrm{F}_{0,\alpha}$, where the branch locus B_α for $\mathrm{F}_{0,\alpha}$ is

$$u(u^2y^3+z^5+\alpha u^4 z).$$

Now $\mathrm{F}_{0,\alpha}$ have singularities over $u=z=0$ of identical type for all α, hence one may resolve these, and obtain a family of surfaces $\mathrm{F}_{1,\alpha}$ containing F_1. But since B_α, for general α, has no singularity except $u=z=0$, the general $\mathrm{F}_{1,\alpha}$ is non-singular. Now by the invariance of p_a [21], $p_a(\mathrm{F}_1)=p_a(\mathrm{F}_{1,\alpha})\leqslant \dim \mathrm{H}^2(\mathrm{F}_{1,\alpha},\Omega)=\dim \mathrm{H}^0(\mathrm{F}_{1,\alpha},\Omega(\mathrm{K}))$, K the canonical class on $\mathrm{F}_{1,\alpha}$. But if ω is the double *quadratic* differential (i.e. of type $\mathrm{A}(dx_\wedge dy)^2$ locally) on P^2 with poles exactly at B_α, one can readily compute $(f_\alpha^*\omega)$, where $f_\alpha\colon \mathrm{F}_{1,\alpha}\to \mathrm{P}^2$; it turns out strictly negative, and as it represents 2 K, it follows

$$p_g(\mathrm{F}_{1,\alpha})=\dim \mathrm{H}^0(\mathrm{F}_{1,\alpha},\Omega(\mathrm{K}))=0.$$

For details on the behaviour of p_a of double planes, which include our result as a particular case, see the works of Enriques and Campedelli cited in [4], p. 203-4, and the doctoral thesis of M. Artin [Harvard, 1960].

REFERENCES

[1] H. Behnke and H. Grauert, Analysis in non-compact Spaces, in *Analytic Functions*, Princeton, 1960.

[2] A. Borel and J.-P. Serre, Le Théorème de Riemann-Roch, *Bull. Soc. Math. France, 86*, 1958.

[3] K. Brauner, Klassifikation der Singularitâten Algebroiden Kurven, *Abh. Math. Semin. Hamburg, 6*, 1928.

[4] F. Enriques, *Le Superficie Algebriche*, Zanichelli, Bologna, 1949.

[5] A. Kurosch, *Theory of Groups*, vol. 2, Chelsea, N.Y.

[6] T. Matsusaka, On Algebraic Familes of Positive Divisors, *J. Math. Soc. Japan, 5*, 1953.

[7] P. Samuel, Multiplicités de certaines composantes singulières, *Ill. J. Math., 3*, 1959.

[8] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, 1934.

[9] H. Seifert and W. Threlfall, *Variationsrechnung im Grossen*, Chelsea, 1951.

[10] S. Smale, The Classification of Immersions of Spheres in Euclidean Space, *Annals Math., 69*, 1959.

[11] R. Thom, *Forthcoming paper on the differential structures on cells and spheres.*

[12] H. Weyl, *Algebraic Theory of Numbers*, Princeton, 1940.

[13] J. H. C. Whitehead, Simplicial Spaces, Nuclei, and m-Groups, *Proc. London Math. Soc., 45*, 1939.

[14] O. Zariski, On the Topology of Algebroid Singularities, *Am. J. Math., 54*, 1932.

[15] O. Zariski, The Reduction of Singularities of an Algebraic Surface, *Annals Math., 40*, 1939.

[16] O. Zariski, Foundations of a General Theory of Birational Correspondences, *Trans. Am. Math. Soc., 53*, 1943.

[17] O. Zariski, Theory and Applications of Holomorphic Functions on an Algebraic Variety of Arbitrary Characteristic, *Memoirs of the Am. Math. Soc.*, 1951.

[18] O. Zariski, Introduction to the Problem of Minimal Models, *J. Math. Soc. Japan*, 1958.

[19] J. E. Reeve, A Note on Fractional Intersection Multiplicities, *Rend. Circolo Mat. Palermo, 1*, 1958.

[20] H. Seifert, Topologie Dreidimensionaler Gefaserter Räume, *Acta Math., 60*, 1932.

[21] J. Igusa, Arithmetic Genera of Normal Varieties in an Algebraic Family, *Proc. Nat. Acad. Sci., 41*, 1955.

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(1) This may be seen by means of a suitable resolution of $(\mathrm{R}^0 f)(\Omega_{\mathrm{G}})$, $f\colon \mathrm{G}\to \mathrm{P}^2$ being its double covering. It is, however, classical: cf. [4], p. 180-2 using the formula:

$$4p_a=n+\mathrm{P}-3\pi-k/3-2 \text{ where } n=2,\ k=0,$$
$$\pi=m-1, \text{ and}$$
$$\mathrm{P}=(2m-1)(2m-2)/2=p_a(\text{Branch Locus}).$$

CHARACTERS AND COHOMOLOGY OF FINITE GROUPS

By M. F. ATIYAH

A N. Bourbaki pour son n^e anniversaire.

INTRODUCTION

For any finite group G one can consider the integral cohomology ring $H^*(G, \mathbf{Z})$ and the ring $R(G)$ of unitary characters (cf. § 6). For 1-dimensional characters there is a well-known isomorphism

$$H^2(G,\mathbf{Z}) \cong \mathrm{Hom}\,(G, U(1)).$$

The purpose of this paper is to establish a connection between the rings $H^*(G, \mathbf{Z})$ and $R(G)$ which, in a certain sense, provides a generalization of the above isomorphism. We shall prove that there is a spectral sequence $\{E_r^p\}$ with

$$E_2^p = H^p(G, \mathbf{Z})$$
$$E_\infty^p = R_p(G)/R_{p+1}(G),$$

where $R(G) = R_0(G) \supset \ldots \supset R_p(G) \supset R_{p+1}(G) \supset \ldots$ is a certain filtration of $R(G)$. This spectral sequence has the following additional properties.

a) A homomorphism $G \to G'$ induces a homomorphism of spectral sequences $E'_r \to E_r$.

b) A monomorphism $G \to G'$ induces a homomorphism of spectral sequences $E_r \to E'_r$ (compatible with the transfer and induced representations).

c) There is a product structure compatible with the products in $H^*(G, \mathbf{Z})$ and $R(G)$.

d) All the even operators d_{2r} are zero.

e) The filtration of $R(G)$ is even, i.e. $R_{2k-1}(G) = R_{2k}(G)$.

It follows from *d)* and *c)* that, if $H^q(G, \mathbf{Z}) = 0$ *for all odd q*, $H^*(G, \mathbf{Z})$ is isomorphic to the graded ring of $R(G)$. This applies notably to the Artin-Tate groups (with periodic cohomology).

The filtration on $R(G)$ has one further property, which we proceed to describe. Let $\varepsilon: R(G) \to \mathbf{Z}$ be the homomorphism obtained by assigning to each character its value at the identity of G, and let $I(G)$ be the kernel of ε. Then we have:

f) The filtration topology of $R(G)$ coincides with its $I(G)$-adic topology.

In view of *f)* the $I(G)$-adic completion $\widehat{R(G)}$ of $R(G)$ plays an important role throughout.

The filtration on $R(G)$ and the spectral sequence are defined topologically, and the whole paper rests heavily on the fundamental results of Bott [3, 4] on the homotopy

of the unitary group. The basic idea may be described as follows. Associated with the group G there is its classifying space B_G, i.e. a space with fundamental group G and contractible universal covering space. The homotopy type of B_G is uniquely determined by G, so that homotopy invariants of B_G give rise to algebraic invariants [1] of G. The most obvious example is provided by the cohomology ring. Now in [1] a homotopy invariant ring $K^*(X)$ was introduced for a finite CW-complex X. This was constructed from complex vector bundles over X, the addition and multiplication in $K^*(X)$ being induced by the direct sum and tensor product of vector bundles. Although B_G is not a finite complex, it is a limit of finite complexes and so [2] $\mathscr{K}^*(B_G)$ can be defined as an inverse limit. Because a representation of G induces a vector bundle over B_G we have a natural ring homomorphism

$$\alpha : R(G) \to \mathscr{K}^*(B_G).$$

The essential content of this paper is the study of the homomorphism α. The main result is that α induces an isomorphism

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G),$$

where $\widehat{R(G)}$ is the I(G)-adic completion of R(G) as above. This identifies the *algebraic invariant* $\widehat{R(G)}$ with the *homotopy invariant* $\mathscr{K}^*(B_G)$.

The spectral sequence relating $H^*(G, \mathbf{Z})$ and $\widehat{R(G)}$ now follows from a spectral sequence $H^*(X, \mathbf{Z}) \Rightarrow K^*(X)$ for any finite CW-complex X [1]. Actually this step involves an inverse limit process and has to be treated with care.

The spectral sequence which we have been discussing for the group G is a special case of a more general "Hochschild-Serre" spectral sequence for a normal subgroup V of a group G. This has

$$E_2^p = H^p(S, \widehat{R(V)}),$$
$$E_\infty^p = R_p(G)_S / R_{p+1}(G)_S,$$

where $S = G/V$ operates on $\widehat{R(V)}$ by conjugation, and

$$R(G) = R_0(G)_S \supset \dots \supset R_p(G)_S \supset \dots$$

is a filtration on R(G) defined relative to S. This reduces to the previous spectral sequence on taking V to be the identity.

The layout of the paper is as follows. In § 1 we discuss vector bundles and representations. In § 2 we summarize the theory of the ring $K^*(X)$. In § 3 we collect together a number of results on inverse limits and completions which will be needed later. Then in § 4 we extend the theory of § 2, with suitable restrictions, to infinite dimensional complexes. The results of § 4 are applied in § 5 to the classifying space of a finite group. The main result (5.1) asserts the existence of a strongly convergent spectral sequence

$$H^*(S, \mathscr{K}^*(B_V)) \Rightarrow \mathscr{K}^*(B_V)_S$$

---

(1) These are necessarily invariant under conjugation, since this just corresponds to a change of base point in B_G.
(2) As in [1] we use $\mathscr{K}$ for the inverse limit K.

where V is normal in G and $S = G/V$. At this point the topological side of the problem is essentially completed, and we turn in § 6 to a study of the ring R(G) with its I(G)-adic topology. The main result of this section (6.1) asserts that the I(H)-adic topology of R(H) is the same as its I(G)-adic topology (for $H \subset G$). This is a most important property of this topology and it leads to a number of basic results for the completion $\widehat{R(G)}$. We also identify the kernel of the homomorphism $R(G) \to \widehat{R(G)}$, showing that it is not in general zero, i.e. that the topology of R(G) is not Hausdorff, but that it *is* zero if G is a p-group.

In § 7 we enunciate the main theorems in a precise form. The next four sections are devoted to the proof of the isomorphism of $\hat{\alpha}: \widehat{R(G)} \to \mathscr{K}^*(B_G)$. The case of a cyclic group G is dealt with explicitly in § 8, and the fact that $\hat{\alpha}$ has zero kernel for general G is shown to follow. In § 9 we digress to establish a few simple lemmas on representations. In § 10 we show that $\hat{\alpha}$ is an isomorphism for solvable groups by using an induction argument based on the spectral sequence of (5.1). The results of § 9 are needed at this stage of the proof. Finally in § 11 the main theorem is extended from solvable groups to general groups by using the "completion" of Brauer's theorem [5] on the characters of finite groups.

An important problem which is left outstanding is that of giving an algebraic description of the filtration on R(G). For cyclic groups this is solved by (8.1), and the case of a general group can be reduced to that of p-groups by (4.9). In § 12 we consider a certain algebraic filtration which has been introduced by Grothendieck. One may conjecture that this coincides with our filtration on R(G). In § 13 we compute some illustrative examples.

This paper seems the appropriate place to point out that a representation of a finite group has certain cohomological invariants called Chern classes (1). In an appendix we summarize their formal properties and discuss their relation with our spectral sequence.

This paper is based on the joint work of F. Hirzebruch and the author, and much of its content was in fact worked out jointly. The corresponding theory for compact connected Lie groups will be found in [1]. It seems likely that the results of this paper and those of [1] are extreme cases of a theorem valid for arbitrary compact Lie groups (2).

On the algebraic side I am greatly indebted to J. Tate and J.-P. Serre for their generous help, without which this paper would not have materialized. This applies in particular to the important § 6.

§ 1. Vector bundles and representations.

For general definitions and properties of fibre bundles we refer to [2], [9] and [12]. We recall that if ξ is a principal bundle over a space X with group G, and if $\rho: G \to H$

(1) This is of course well-known to topologists.
(2) (Added in proof). This is in fact the case. It will be dealt with in a separate publication.

is a homomorphism, then we have an induced principal bundle over X with group H, which is denoted by $\rho(\xi)$ [2, § 6]. We shall be concerned with the case when ξ is the universal covering space of X, so that $G=\pi_1(X)$ is the fundamental group of X. Moreover we shall suppose that G is finite. For H we take the general linear group $GL(n, \mathbf{C})$, and we shall consider this *a)* with the discrete topology, and *b)* with its ordinary topology. The corresponding principal bundles will be called *discrete* $GL(n, \mathbf{C})$-bundles or *ordinary* $GL(n, \mathbf{C})$-bundles according as we use topology *a)* or topology *b)*. For the discrete case we have [1] [12, § 13.9]:

Proposition (**1.1**). — *The mapping* $\rho\to\rho(\xi)$ *sets up a* (1—1) *correspondence between the equivalence classes of unitary representations of* G *of degree n and the isomorphism classes of discrete* $GL(n, \mathbf{C})$*-bundles over* X.

The purpose of (1.1) is simply to translate representations into a geometrical form.

From any $GL(n, \mathbf{C})$-bundle (discrete or ordinary) one can form the associated n-dimensional complex vector bundle over X, and conversely given the vector bundle the principal bundle may be recovered as the bundle of n-frames. We proceed to translate (1.1) into terms of vector bundles. Let E be a complex representation space of G (or G-module). Then we may form the vector bundle $E(\xi)$ over X associated to ξ. $E(\xi)$ may be considered either as an ordinary vector bundle or as a discrete vector bundle according as E is taken with the ordinary or the discrete topology. Then from (1.1) we have.

Proposition (**1.2**). — *The mapping* $E\to E(\xi)$ *sets up a* (1—1) *correspondence between the isomorphism classes of complex* G*-modules and the isomorphism classes of discrete complex vector bundles over* X.

For G-modules, discrete vector bundles and ordinary vector bundles one has the following operations and maps.

1) *Direct sum* $E\oplus F$;
2) *Tensor product* $E\otimes F$;
3) *Exterior powers* $\lambda^i(E)$;
4) *Inverse image* f^*E;
5) *Direct image* f_*E.

1), 2) and 3) need no explanation (for vector bundles see [9, § 3.6]). 4) is to be understood as follows. If f: $H\to G$ is homomorphism of groups, and E is a G-module, then E is also an H-module and as such is denoted by f^*E. If $f: Y\to X$ is a continuous map of spaces, and E is a (discrete or ordinary) vector bundle over X, then f^*E is the induced vector bundle over Y. 5) is defined when f is a monomorphism in the group case or a finite covering in the space case. For groups f_*E is the induced representation module, and for coverings f_*E is the direct image bundle, i.e. the fibre $(f_*E)_x$ is defined as the direct sum $\oplus_y E_y$ where $y\in f^{-1}(x)$.

[1] We suppose X satisfies the requirements of [12, § 13.9]. For example we could take X a finite CW-complex.

It is not difficult to check that 1)-5) are compatible with the mapping of (1.2) and also with the passage from discrete vector bundles to ordinary vector bundles. We observe only that if $\pi_1(X)=G$, $\pi_1(Y)=H$, a map f induces a homomorphism $H\to G$ and that this is a monomorphism if f is a finite covering.

1)-5) have the following properties:

a) $\oplus$ is commutative and associative;

b) $\otimes$ is associative and distributive over $\oplus$;

c) f^* commutes with $\oplus$, $\otimes$ and λ^i;

d) f_* commutes with $\oplus$;

e) $f_*(E\otimes f^*F)\cong f_*(E)\otimes F$.

These follow trivially from the definitions. For representations *e)* is the so called reciprocity formula.

Our main problem is to study the passage from representations ρ of G to the ordinary $GL(n, \mathbf{C})$-bundle $\rho(\xi)$. In view of (1.1) and using the notation of [9, § 3.1], this means we have to study the map

$$H^1(X, GL(n, \mathbf{C}))\to H^1(X, GL(n, \mathbf{C})_c)$$

where $GL(n, \mathbf{C})$ denotes the constant sheaf and $GL(n, \mathbf{C})_c$ denotes the sheaf of germs of continuous maps $X\to GL(n, \mathbf{C})$. In general this cohomology formulation of the problem is of no help, but when $n=1$, the sheaves are sheaves of *abelian* groups and the problem can be dealt with as follows.

We have two exact sequences of sheaves, related by homomorphisms:

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z} & \longrightarrow & \mathbf{C} & \xrightarrow{\exp 2\pi i} & \mathbf{C}^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z} & \longrightarrow & \mathbf{C}_c & \longrightarrow & \mathbf{C}^*_c & \to & 0 \end{array}$$

where $\mathbf{C}^*=GL(1, \mathbf{C})$. These give cohomology exact sequences (cf. [9, § 3.8])

$$\begin{array}{ccccccccc} \to H^1(X, \mathbf{C}) & \to & H^1(X, \mathbf{C}^*) & \to & H^2(X, \mathbf{Z}) & \to & H^2(X, \mathbf{C}) & \to \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to H^1(X, \mathbf{C}_c) & \to & H^1(X, \mathbf{C}^*_c) & \to & H^2(X, \mathbf{Z}) & \to & H^2(X, \mathbf{C}_c) & \to \end{array}$$

Now the sheaf $\mathbf{C}_c$ is fine and so $H^q(X, \mathbf{C}_c)=0$ $(q>0)$ [9, § 2.11]. Hence we deduce:

Proposition **(1.3)**. — *Let* X *have zero Betti numbers in dimensions* 1 *and* 2. *Then we have canonical isomorphisms*:

$$\mathrm{Hom}(\pi_1(X), \mathbf{C}^*)\cong H^1(X, \mathbf{C}^*)\cong H^1(X, \mathbf{C}^*_c)\cong H^2(X, \mathbf{Z}).$$

In (1.3) we may take X to be the 3-skeleton of the classifying space of B_G, where G is finite. We obtain the isomorphism

$$\mathrm{Hom}(G, \mathbf{C}^*)\cong H^2(X, \mathbf{Z})$$

referred to in the introduction (with $\mathbf{C}^*$ replacing $U(1)$), and which it is our purpose to generalize.

The notion of discrete vector bundles was introduced in this section simply as a bridge between representations and ordinary vector bundles. From now on all vector bundles will be ordinary.

§ 2. $K^*(X)$ for finite-dimensional X.

We propose here to recall briefly the definition and basic properties of the functor K^* introduced in [1]. All spaces considered in this section will be finite CW-complexes (cf. [13]).

Let $F(X)$ be the free abelian group generated by the set of all isomorphism classes of complex vector bundles over X. To every triple $t=(\xi, \xi', \xi'')$ of vector bundles with $\xi \simeq \xi' \oplus \xi''$ we assign the element $[t]=[\xi]-[\xi']-[\xi'']$ of $F(X)$, where $[\xi]$ denotes the isomorphism class of ξ. The group $K^0(X)$ is defined as the quotient of $F(X)$ by the subgroup generated by all the elements of the form $[t]$. In this definition we allow a vector bundle to have different dimensions over the different connectedness components of X.

The tensor product of vector bundles defines a commutative ring structure in $K^0(X)$; the unit 1 is given by the trivial bundle of dimension one. $K^0(X)$ is a contravariant functor of X.

Let S^1 denote the circle and let $X \to X \times S^1$ be the embedding given by a base point of S^1. We define $K^1(X)$ to be the kernel of the induced homomorphism

$$K^0(X \times S^1) \to K^0(X),$$

and we put $K^*(X)=K^0(X) \oplus K^1(X)$. The ring structure on $K^0(X)$ extends to give a ring structure on $K^*(X)$, and $K^*(X)$ is again a contravariant functor. Moreover it is an invariant of homotopy type. A map $f: Y \to X$ induces a homomorphism $K^*(X) \to K^*(Y)$ which will be denoted by $f^!$. For a point we have:

$$K^0(\text{point}) \simeq \mathbf{Z}, \quad K^1(\text{point})=0. \tag{2.1}$$

For a connected space X the fibre dimension defines an "augmentation" $\varepsilon: K^0(X) \to \mathbf{Z}$. In view of (2.1) this is the restriction to $K^0(X)$ of the homomorphism $i^!: K^*(X) \to K^*(\text{point})$ induced by the inclusion of a point in X. Using $i^!$ we extend ε to $K^*(X)$. If we denote the kernel of ε by $\widetilde{K}^*(X)$ there is a canonical decomposition $K^*(X) \simeq \widetilde{K}^*(X) \oplus \mathbf{Z}$.

We define a filtration on $K^*(X)$ by putting $K_p^*(X)=\operatorname{Ker}\{K^*(X) \to K^*(X^{p-1})\}$, where X^{p-1} is the $(p-1)$-skeleton of X. If X is connected $K_1^*(X)=\widetilde{K}^*(X)$. This filtration is a homotopy invariant and turns $K^*(X)$ into a filtered ring, i.e.

$$K_p^*(X) \,.\, K_q^*(X) \subset K_{p+q}^*(X).$$

It has moreover the following property:

$$K_{2k-1}^0(X)=K_{2k}^0(X), \quad K_{2k}^1(X)=K_{2k+1}^1(X). \tag{2.2}$$

If $f: Y \to X$ is a finite covering the direct image of vector bundles (cf. § 1) induces a group homomorphism $f_!: K^0(Y) \to K^0(X)$. Replacing X, Y by $X \times S^1, Y \times S^1$ this extends to a homomorphism $f_!: K^*(Y) \to K^*(X)$ which preserves filtration (since we may take $Y^{p-1} = f^{-1}(X^{p-1})$ [13; § 4, 5]), multiplies the augmentation by the degree of f and satisfies the formula

(2.3) $$f_!(y.f^!(x)) = f_!(y).x, \quad y \in K^*(Y), \quad x \in K^*(X).$$

For elements of K^0 this formula follows at once from *e)* of § 1. The general case can then be shown to follow [1].

We come now to the most important property for our present purposes, the existence of the spectral sequence. We state this as a proposition.

Proposition (2.4). — *There is a spectral sequence* $\{E^p(X)\}$ *with* $E_2^p(X) = H^p(X, \mathbf{Z})$, $E_\infty^p(X) = K_p^*(X)/K_{p+1}^*(X)$, *and with the following further properties.*

a) A map $f: Y \to X$ *induces a homomorphism of spectral sequences* $E_r^p(X) \to E_r^p(Y)$ *which depends only on the homotopy class of f.*

b) A finite covering $f: Y \to X$ *induces a homomorphism of spectral sequences* $E_r^p(Y) \to E_r^p(X)$.

c) The cup-product in $H^*(X, \mathbf{Z})$ *induces products in each* E_r $(2 \leqslant r \leqslant \infty)$ *which for* $r = \infty$ *coincide with the products induced by the ring structure of* $K^*(X)$.

d) The even differentials d_{2r} *are all zero,* d_3 *is the Steenrod operation* Sq^3, *and* $d_r(x) = 0$ *for* $\dim x \leqslant 2$ *and all r.*

Remark. — It is understood of course that the homomorphism of *a)* is compatible with f^* and $f^!$ while that of *b)* is compatible with f_* (the direct image or trace for cohomology) and $f_!$.

In view of the last part of *d)* we have an isomorphism:

(2.5) $$K_2^*(X)/K_3^*(X) \cong H^2(X, \mathbf{Z}).$$

This isomorphism can be described directly as follows. First we observe, using (2.2), that $$K_2^*(X)/K_3^*(X) \cong K_1^0(X)/K_3^0(X).$$

Now by assigning to each vector bundle E over X the 1-dimensional bundle det(E) (i.e. $\lambda^n(E)$ if E has dimension n), and then using (1.3) we obtain a homomorphism (the first Chern class)

$$c_1: K^0(X) \to H^2(X, \mathbf{Z}).$$

Restricting to $K_1^0(X)$ we obtain the homomorphism which induces (2.5).

Next we turn to the more general spectral sequence for a fibre bundle $\pi: Y \to X$ with fibre F. First we define a filtration on $K^*(Y)$ relative to X by putting

$$K_p^*(Y)_X = \operatorname{Ker}\{K^*(Y) \to K^*(Y^{p-1})\}$$

where $Y^{p-1} = \pi^{-1}(X^{p-1})$. Then we have:

Proposition (2.6). — *There is a spectral sequence* $\{E_r^p\}$ *with* $E_2^p = H^p(X, \mathbf{K}^*(\mathbf{F}))$, $E_\infty^p = K_p^*(Y)_X/K_{p+1}^*(Y)_X$, *and with the following further properties.*

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(1) Statements given without proof here or in [1] will be proved in a future publication with F. Hirzebruch.

a) *A commutative diagram*

$$\begin{array}{ccc} Y & \rightarrow & Y' \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi'} \\ X & \rightarrow & X' \end{array}$$

gives rise to a homomorphism of spectral sequences $E'_r \rightarrow E_r$;

b) *The cup-product in* $H^*(X, \mathbf{K}^*(\mathbf{F}))$, *using the ring structure of* $\mathbf{K}^*(\mathbf{F})$, *induces products in each* E_r $(2 \leqslant r \leqslant \infty)$ *which for* $r = \infty$ *coincide with the products induced by the ring structure of* $K^*(Y)$.

c) *If* $K^1(F) = 0$, *all* $d_{2r} = 0$.

Remarks. — 1) Taking F to be a point and $Y = X$, (2.6) reduces to (2.4).

2) $\mathbf{K}^*(\mathbf{F})$ denotes the local coefficient system whose group at x is $K^*(\pi^{-1}(x))$.

3) In *a)* we do not insist that $F = F'$.

Taking $Y' = X' = X$ in *a)*, with $Y \rightarrow Y'$ being π, we get a homomorphism of the spectral sequence of (2.4) into that of (2.6). Hence from (2.6) *b)* we deduce:

(2.7) *The* E_r *of* (2.6) *are modules over the* E_r *of* (2.4).

Applying (2.4) with $X =$ point we get the trivial spectral sequence $\mathbf{Z}$, i.e. $E_r^0 = \mathbf{Z}$, $E_r^p = 0$ for $p > 0$ (all r). Since the spectral sequence of $\pi : y_0 \rightarrow x_0$ with $y_0 \in Y$ and $x_0 = \pi(y_0) \in X$ is a direct factor of that of (2.6) we deduce:

(2.8) *Let* $Y \rightarrow X$ *be a fibre bundle with* Y *and* F *(the fibre) connected. Then the spectral sequence* E_r *of* (2.6) *decomposes*: $E_r = \widetilde{E}_r \oplus \mathbf{Z}$, *where* $\mathbf{Z}$ *denotes the trivial spectral sequence*, $\widetilde{E}_r^p = E_r^p$ *for* $p > 0$ *and* $\widetilde{E}_2^0 = H^0(X, \widetilde{\mathbf{K}}^*(\mathbf{F}))$, $\widetilde{E}_\infty^0 = \widetilde{K}_0^*(Y)_X / \widetilde{K}_1^*(Y)_X$.

Easy consequences of (2.4) are the following:

(2.9) $K^*(X)$ *is a finitely-generated group.*

(2.10) *Let* $f : Y \rightarrow X$ *be such that* $f^* H^q(X, \mathbf{Z})$ *is finite for all* $q > 0$ *and suppose* X *connected. Then* $f^! \widetilde{K}^*(X)$ *is finite.*

We now make one formal application of the properties of $K^*(X)$.

Proposition **(2.11)**. — *Let* $f : Y \rightarrow X$ *be a finite covering of degree d. Then d annihilates the kernel of* [1] $GK^*(X) \rightarrow GK^*(Y)$. *If p is a prime not dividing d, then the p-primary component of* $GK^*(X)$ *is a direct factor of* $GK^*(Y)$.

Proof. — Since $f_!$ and $f^!$ preserve filtration they induce homomorphisms $\varphi_!$ and $\varphi^!$ of GK^*, and from (2.3) we deduce

$$\varphi_!(y \cdot \varphi^!(x)) = \varphi_!(y) \cdot x \qquad x \in GK^*(X), y \in GK^*(Y).$$

Taking $y = 1$ we obtain

$$\varphi_! \varphi^!(x) = \varphi_!(1) \cdot x = dx.$$

The proposition follows at once from this formula.

Remark. — This is quite analogous to the corresponding result for cohomology.

[1] If $A = A_0 \supset A_1 \supset A_2 \supset \ldots$ is a filtered group we denote by GA the graded group $\Sigma_p A_p / A_{p+1}$. The component A_p / A_{p+1} will be denoted by $G^p A$.

§ 3. Inverse limits and completions.

Let M be a filtered abelian group, i.e. we have a sequence of subgroups:

$$M = M_0 \supset M_1 \supset \ldots \supset M_n \supset \ldots$$

This filtration gives M the structure of a topological group, the subgroups M_n being taken as a fundamental system of neighbourhoods of 0 in M. We denote by $\hat{M}$ (or $M^\wedge$) the completion of M for this topology, i.e.

(3.1) $$\hat{M} = \varprojlim M/M_n \text{ (inverse limit)}.$$

We remark that the topology of M is not necessarily Hausdorff so that the natural map $M \to \hat{M}$ may have non-zero kernel. In fact we have:

(3.2) $$\operatorname{Ker}(M \to \hat{M}) = \bigcap_{n=1}^{\infty} M_n.$$

If $\{{}_nA\}$ is an inverse system of abelian groups (indexed by the non-negative integers), the inverse limit $A = \varprojlim {}_nA$ has a natural filtration defined by (1)

$$A_n = \operatorname{Ker}\{A \to {}_{n-1}A\}$$

Moreover A is complete for the topology defined by this filtration, i.e. $A \cong \hat{A}$. Thus an inverse limit is in a natural way a *complete filtered group*. This applies in particular to the group $\hat{M}$ given by (3.1). It is easy to see that the subgroups of the filtration may be identified with the completions $\hat{M}_n$ of the subgroups M_n (for the induced topology).

If M is a finite group then the filtration necessarily terminates, i.e. $M_n = M_{n+1}$ for all $n \geqslant n_0$, and so $\hat{M} \cong M/M_{n_0}$. We record this for future reference.

Lemma (3.3). — *If* M *is a finite filtered group* $M \to \hat{M}$ *is an epimorphism.*

We also state the following elementary properties of inverse limits, the verifications being trivial.

Lemma (3.4). — *Let* $\{{}_{\alpha,\beta}A\}$ *be an inverse system indexed by pairs* $(\alpha, \beta) \in I \times J$, *where* I, J *are two directed sets. Then*

$$\varprojlim_{\alpha} \varprojlim_{\beta} {}_{\alpha,\beta}A \cong \varprojlim_{(\alpha,\beta)} {}_{\alpha,\beta}A \cong \varprojlim_{\beta} \varprojlim_{\alpha} {}_{\alpha,\beta}A$$

Lemma (3.5). — *If* $0 \to \{{}_\alpha A\} \to \{{}_\alpha B\} \to \{{}_\alpha C\} \to 0$ *is an exact sequence of inverse systems* (α *belonging to some directed set*), *then*

$$0 \to \varprojlim {}_\alpha A \to \varprojlim {}_\alpha B \to \varprojlim {}_\alpha C$$

is exact.

In order for $\varprojlim$ to be right exact we need a condition. Following Dieudonné-Grothendieck [8] we adopt the following definition. An inverse system $\{{}_\alpha A\}$ is said to satisfy the *Mittag-Leffler* condition (ML) if, for each α, there exists $\beta \geqslant \alpha$ such that

$$\operatorname{Im}({}_\beta A \to {}_\alpha A) = \operatorname{Im}({}_\gamma A \to {}_\alpha A)$$

(1) We put ${}_{-1}A = 0$ so that $A_0 = A$.

for all $\gamma \geqslant \beta$. Moreover we shall assume from now on that all inverse systems are over *countable* directed sets. The following properties of (ML) are proved in [8, chapter o (complements)].

(3.6) *If* $\{_\alpha A\} \to \{_\alpha B\} \to 0$ *is exact and* $\{_\alpha A\}$ *satisfies* (ML), *so does* $\{_\alpha B\}$.

(3.7) *If* $0 \to \{_\alpha A\} \to \{_\alpha B\} \to \{_\alpha C\} \to 0$ *is exact, and if* $\{_\alpha A\}$ *and* $\{_\alpha C\}$ *each satisfy* (ML), *so does* $\{_\alpha B\}$.

(3.8) *If* $0 \to \{_\alpha A\} \to \{_\alpha B\} \to \{_\alpha C\} \to 0$ *is exact, and if* $\{_\alpha A\}$ *satisfies* (ML), *then*

$$0 \to \varprojlim {}_\alpha A \to \varprojlim {}_\alpha B \to \varprojlim {}_\alpha C \to 0$$

is exact.

(3.9) *Let* $\{_\alpha C^*\}$ *be an inverse system of complexes, with differentials of degree r. Suppose that, for each p,* $\{_\alpha C^p\}$ *and* $\{H^p(_\alpha C^*)\}$ *satisfy* (ML), *then* $\varprojlim H^p(_\alpha C^*) \cong H^p(\varprojlim {}_\alpha C^*)$.

Remark. — In [8] the differentials in (3.9) are supposed to have degree 1, but this does not affect the argument.

Using (3.8) we now prove two further lemmas on completions [1].

Lemma **(3.10)**. — *Let* $f: M \to N$ *be a homomorphism of filtered groups. Then the following two statements are equivalent:*

(i) $\hat{f}: \hat{M} \to \hat{N}$ *is an isomorphism of filtered groups.*

(ii) $Gf: GM \to GN$ *is an isomorphism* (*where* GM *denotes the graded group of* M).

Proof. — Suppose first that (ii) holds. Then $M/M_0 = N/N_0 = 0$ and from the diagram:

$$\begin{array}{ccccccccc} 0 \to & M_n/M_{n+1} & \to & M/M_{n+1} & \to & M/M_n & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & N_n/N_{n+1} & \longrightarrow & N/N_{n+1} & \longrightarrow & N/N_n & \to 0 \end{array}$$

we deduce, by induction on n, that $M/M_n \to N/N_n$ is an isomorphism for all n. Taking inverse limits we deduce (i).

Conversely let (i) hold, then $G\hat{M} \to G\hat{N}$ is an isomorphism. To prove (ii) it will be sufficient therefore to prove that $GM \cong G\hat{M}$, i.e. to prove the special case where $N = \hat{M}$ and f is the natural map. Now we have an exact sequence:

$$0 \to M_{n+1}/M_{n+k} \to M_n/M_{n+k} \to M_n/M_{n+1} \to 0,$$

and the inverse system $\{M_{n+1}/M_{n+k}\}$ (for n fixed and $k \to \infty$) satisfies (ML) trivially since all maps are epimorphisms. Hence by (3.8) we deduce the exact sequence:

$$0 \to \hat{M}_{n+1} \to \hat{M}_n \to M_n/M_{n+1} \to 0,$$

which proves that $GM \cong G\hat{M}$ as required.

Lemma **(3.11)**. — *Let* $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ *be an exact sequence of abelian groups. Let* M_n *be a filtration of* M *and define filtrations of* M′, M″ *by* $M'_n = \alpha^{-1}(M_n)$, $M''_n = \beta(M_n)$. *Then* $0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$ *is exact.*

[1] Direct proofs are also possible.

Proof. — From the definitions of M'_n, M''_n it follows that

$$0 \to M'/M'_n \to M/M_n \to M''/M''_n \to 0$$

is exact. Since the inverse system $\{M'/M'_n\}$ satisfies (ML) the lemma follows from (3.8).

In addition to the condition (ML) we shall require later the following:

(F) *For each* α *there exists* $\beta \geqslant \alpha$ *so that* $\mathrm{Im}({}_\beta A \to {}_\alpha A)$ *is finite.*

Clearly (F) implies (ML).

Lemma (**3.12**). — *Let* $0 \to \{{}_\alpha A\} \to \{{}_\alpha B\} \to \{{}_\alpha C\} \to 0$ *be an exact sequence of inverse systems and suppose* $\{{}_\alpha B\}$ *satisfies* (F). *Then* $\{{}_\alpha A\}$ *and* $\{{}_\alpha C\}$ *also satisfy* (F).

Proof. — By hypothesis there exists $\beta \geqslant \alpha$ so that ${}_{\beta,\alpha}B = \mathrm{Im}({}_\beta B \to {}_\alpha B)$ is finite. Define ${}_{\beta,\alpha}A$, ${}_{\beta,\alpha}C$ similarly, then we have exact sequences

$${}_{\beta,\alpha}B \to {}_{\beta,\alpha}C \to 0, \qquad 0 \to {}_{\beta,\alpha}A \to {}_{\beta,\alpha}B.$$

These imply that ${}_{\beta,\alpha}C$ and ${}_{\beta,\alpha}A$ are finite. Q.E.D.

Lemma (**3.13**). — *Let* $\{{}_\alpha A\}$ *satisfy* (F), *then* $\varprojlim {}_\alpha A$ *is a compact Hausdorff group.*

Proof. — Let ${}_\alpha B = \bigcap_{\beta \geqslant \alpha} \mathrm{Im}({}_\beta A \to {}_\alpha A)$. Then (F) implies that ${}_\alpha B$ is finite. But, from the definition of inverse limits, $\varprojlim {}_\alpha A \cong \varprojlim {}_\alpha B$. Thus $\varprojlim {}_\alpha A$ is an inverse limit of finite groups and so compact and Hausdorff (for the inverse limit topology, i.e. the topology induced from the direct product $\Pi_\alpha A$).

Lemma (**3.14**). — *Let* $\{{}_\alpha A\}$ *be an inverse system indexed by* I, *and let* J *be a cofinal subset of* I. *Then* $\varprojlim_{\alpha \in I} {}_\alpha A \cong \varprojlim_{\alpha \in J} {}_\alpha A$, *and* $\{{}_\alpha A\}_{\alpha \in I}$ *satisfies* (ML) *or* (F) $\Leftrightarrow \{{}_\alpha A\}_{\alpha \in J}$ *satisfies* (ML) *or* (F).

Proof. — The isomorphism of the inverse limits is well-known and the implication $\Rightarrow$ is trivial. Suppose $\{{}_\alpha A\}_{\alpha \in J}$ satisfies (ML), and let $\lambda \in I$. Since J is cofinal there exists $\alpha \in J$, $\alpha \geqslant \lambda$. Since $\{{}_\alpha A\}_{\alpha \in J}$ satisfies (ML) there exists $\beta \in J$, $\beta \geqslant \alpha$ so that $\mathrm{Im}({}_\beta A \to {}_\alpha A) = \mathrm{Im}({}_\gamma A \to {}_\alpha A)$ for all $\gamma \in J$, $\gamma \geqslant \beta$. Now let $\mu \in I$, $\mu \geqslant \beta$ and (J being cofinal) choose $\gamma \in J$, $\gamma \geqslant \mu$. Then it follows that $\mathrm{Im}({}_\beta A \to {}_\lambda A) = \mathrm{Im}({}_\mu A \to {}_\lambda A)$ which shows that $\{{}_\alpha A\}_{\alpha \in I}$ satisfies (ML). If $\{{}_\alpha A\}_{\alpha \in J}$ satisfies (F) then, with the same notation, $\mathrm{Im}({}_\beta A \to {}_\alpha A)$ is finite. This implies that $\mathrm{Im}({}_\beta A \to {}_\lambda A)$ is finite, showing that $\{{}_\alpha A\}_{\alpha \in I}$ satisfies (F).

Noetherian Completions.

Let A be a Noetherian ring (commutative and with identity), let $\mathfrak{a}$ be an ideal in A and let M be a finitely-generated A-module. We define a filtration on M by $M_n = \mathfrak{a}^n M$. The topology defined by this filtration is called the "$\mathfrak{a}$-adic" topology of M or simply the $\mathfrak{a}$-topology of M. This topology has a number of important properties which we proceed to recall (cf. [6, exp. 18]).

Proposition (**3.15**). — *Let* M *be a finitely generated* A*-module and let* N *be a sub-module of* M. *Then the topology of* N *induced by the* $\mathfrak{a}$*-topology of* M *coincides with the* $\mathfrak{a}$*-topology of* N.

Proof. — We have the inclusion

$$\mathfrak{a}^n N \subset \mathfrak{a}^n M \cap N.$$

To prove the proposition therefore it is sufficient to show that, for each n, there exist m so that

1) $$\mathfrak{a}^m M \cap N \subset \mathfrak{a}^n N.$$

But by the lemma of Artin-Rees [6, exp. 2] there exists m_0 such that, for $m \geqslant m_0$,

2) $$\mathfrak{a}^m M \cap N = \mathfrak{a}^{m-m_0}(\mathfrak{a}^{m_0} M \cap N).$$

Taking $m = n + m_0$, 1) follows at once from 2).

Proposition **(3.16)**. — *For finitely-generated* A*-modules,* $\mathfrak{a}$*-adic completion is an exact functor.*

Proof. — Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely-generated A-modules. By (3.15) the $\mathfrak{a}$-topology of M' is induced from the $\mathfrak{a}$-topology of M. Also the $\mathfrak{a}$-topology of M'' is induced from the $\mathfrak{a}$-topology of M. Hence we can apply (3.11) and we deduce the exact sequence $0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$, where each completion is the $\mathfrak{a}$-adic completion.

Let G be a finite group. M will be called an A-G-module if it is both an A-module (finitely-generated) and a G-module, and if the operations of A and G on M commute. Then the cohomology groups $H^q(G, M)$ will be (finitely-generated) A-modules and so we can form their $\mathfrak{a}$-adic completions $H^q(G, M)\hat{}$. On the other hand, since $g(\alpha(m)) = \alpha(g(m))$ for all $\alpha \in A$, $g \in G$, $m \in M$, it follows that the sub-modules $\mathfrak{a}^n M$ are stable under G. Hence G operates on $M/\mathfrak{a}^n M$ and so on $\hat{M}$. Then we have:

Proposition **(3.17)**. — *Let* M *be an* A-G*-module. Then we have a canonical isomorphism*:

$$H^q(G, M)\hat{} \cong H^q(G, \hat{M}).$$

Proof. — Let $\Lambda = \mathbf{Z}[G]$ be the group ring of G and let $\{X_n\}$ be the standard Λ-free resolution of $\mathbf{Z}$ [7, chapter X]. By definition

1) $$H^q(G, M) = H^q(\mathrm{Hom}_\Lambda(X_*, M)).$$

Now $\mathrm{Hom}_\Lambda(X_q, M)$ is, for each q, a finitely-generated A-module. Hence by (3.16):

2) $$H^q(\mathrm{Hom}_\Lambda(X_*, M))\hat{} \cong H^q(\mathrm{Hom}_\Lambda(X_*, M)\hat{}).$$

Since X_q is, for each q, a free Λ-module it follows that

3) $$\mathrm{Hom}_\Lambda(X_*, M)\hat{} \cong \mathrm{Hom}_\Lambda(X_*, \hat{M}).$$

From 1), 2) and 3) the proposition follows.

Spectral sequences.

We propose next to consider inverse limits of spectral sequences. By a spectral sequence we shall understand a sequence of complexes [1] $\{E_r\}$, $2 \leqslant r < \infty$, with given isomorphisms $E_{r+1} \cong H(E_r)$. We suppose that $E_r^p = 0$ for $p < 0$ and that the

[1] I.e. graded abelian groups with a differential (endomorphism d satisfying $d^2 = 0$).

differential d_r of E_r has degree r. Thus each E_r^p, for $2 \leqslant r < \infty$, may be identified with a quotient group Z_r^p/B_r^p, where [1] Z_r^p, B_r^p are subgroups of E_2^p. These subgroups are arranged as follows:

$$(\mathscr{S}) : 0 = B_2^p \subset \ldots \subset B_r^p \subset \ldots \subset \ldots \subset Z_r^p \ldots \subset Z_2^p = E_2^p.$$

We then define

$$B_\infty^p = \bigcup_r B_r^p = \varinjlim_r B_r^p,$$

$$Z_\infty^p = \bigcap_r Z_r^p = \varprojlim_r Z_r^p,$$

$$E_\infty^p = Z_\infty^p / B_\infty^p.$$

By a *strongly convergent spectral sequence* we shall mean a spectral sequence $\{E_r\}$ together with a *complete filtered group* M and isomorphisms $E_\infty^p \simeq M_p/M_{p+1}$. We shall write $E_2 \Rightarrow M$, and say that the spectral sequence $\{E_r\}$ converges strongly to M. This is in agreement with the terminology of [7, chapter XV, § 2]. If all the differentials d_r are zero we shall say that the spectral sequence collapses. The following is then an immediate consequence of the definitions.

Proposition **(3.18)**. — *Suppose* $E_2 \Rightarrow M$ *and the spectral sequence collapses. Then* $E_2 \simeq GM$.

We now prove a result on inverse limits of spectral sequences.

Proposition **(3.19)**. — *Let* $\{{}_nE_r, {}_nM\}$ *be an inverse system of strongly convergent spectral sequences. Suppose further that*

a) *For each* p, $\{{}_nE_2^p\}$ *satisfies* (F);
b) $\{{}_nM\}$ *satisfies* (F).

Then $\{\varprojlim {}_nE_r, \varprojlim {}_nM\}$ *is a strongly convergent spectral sequence.*

Proof. — For each n we have the sequence of inclusions:

$$({}_n\mathscr{S}) : 0 = {}_nB_2^p \subset \ldots \subset {}_nB_r^p \subset \ldots \subset {}_nZ_r^p \subset \ldots \subset {}_nZ_2^p = {}_nE_2^p.$$

Now hypothesis *a)* and (3.12) imply that $\{{}_nE_r^p\} = \{{}_nZ_r^p/{}_nB_r^p\}$ satisfies (F), and so (ML). Hence writing $E_r^p = \varprojlim {}_nE_r^p$ and using (3.9) we see that $E_{r+1}^p \simeq H^p(E_r)$, showing that $\{E_r\}$ $(2 \leqslant r < \infty)$ is indeed a spectral sequence (the operators are of course defined as $d_r = \varprojlim {}_nd^r$).

Taking the inverse limit of $({}_n\mathscr{S})$ we get (by (3.5)) a sequence of inclusions:

$$(\mathscr{S}) \qquad 0 = B_2^p \subset \ldots \subset B_r^p \subset \ldots \subset Z_r^p \subset \ldots \subset Z_2^p = E_2^p,$$

where $B_r^p = \varprojlim {}_nB_r^p$, $Z_r^p = \varprojlim {}_nZ_r^p$. By *a)* and (3.12) again $\{{}_nB_r^p\}$ satisfies (F) and so (ML). Hence by (3.8) $E_r^p \simeq Z_r^p/B_r^p$, showing that $(\mathscr{S})$ has the same significance as before. Now the fact that ${}_nE_r^p = 0$ for $p < 0$ and that d_r has degree r imply that, for all n, ${}_nB_\infty^p \simeq {}_nB_r^p$ for $r \geqslant p+1$.

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[1] This conflicts slightly with the usual notation, but should cause no confusion.

Hence
$$\varprojlim {}_n\mathrm{B}^p_\infty \cong \mathrm{B}^p_r \text{ for } r \geqslant p+1,$$
$$\cong \mathrm{B}^p_\infty.$$
Also
$$\begin{aligned}\varprojlim {}_n\mathrm{Z}^p_\infty &= \varprojlim_n \varprojlim_r {}_n\mathrm{Z}^p_r \\ &\cong \varprojlim_r \varprojlim_n {}_n\mathrm{Z}^p_r \qquad \text{by } (3.4), \\ &= \varprojlim_r \mathrm{Z}^p_r \\ &= \mathrm{Z}^p_\infty.\end{aligned}$$

We now consider $\mathrm{M} = \varprojlim {}_n\mathrm{M}$. In view of (3.5) this is filtered by subgroups $\mathrm{M}_p = \varprojlim {}_n\mathrm{M}_p$. To complete the proof we have to show

(i) $\mathrm{Z}^p_\infty/\mathrm{B}^p_\infty \cong \mathrm{M}_p/\mathrm{M}_{p+1}$;

(ii) $\mathrm{M} \cong \varprojlim \mathrm{M}/\mathrm{M}_p$.

Now $\{{}_n\mathrm{B}^p_\infty\}$ satisfies (F) ((*a*) and (3.12)), and $\{{}_n\mathrm{M}_{p+1}\}$ satisfies (F)) (*b*) and (3.12)). Hence, by (3.8), (i) follows from the corresponding isomorphisms ${}_n$(i) and the isomorphisms $\mathrm{B}^p_\infty \cong \varprojlim {}_n\mathrm{B}^p_\infty$, $\mathrm{Z}^p_\infty \cong \varprojlim {}_n\mathrm{Z}^p_\infty$ established above. Similarly (ii) follows from ${}_n$(ii), using (3.8) and (3.4).

§ 4. $\mathscr{K}^*(\mathrm{X})$ for infinite-dimensional X.

In this section we shall extend the definition and properties of $\mathrm{K}^*(\mathrm{X})$, as given in § 2, to CW-complexes X all of whose skeletons are finite [1]. Throughout this section a CW-complex will always mean one with this property. We define
$$\mathscr{K}^*(\mathrm{X}) = \varprojlim \mathrm{K}^*(\mathrm{X}^n),$$
where X^n is the n-skeleton of X. Then, as remarked in § 3, $\mathscr{K}^*(\mathrm{X})$ is in a natural way a complete filtered group, the filtration being defined by
$$\begin{aligned}\mathscr{K}^*_p(\mathrm{X}) &= \mathrm{Ker}\{\mathscr{K}^*(\mathrm{X}) \to \mathrm{K}^*(\mathrm{X}^{p-1})\} \\ &\cong \varprojlim_n \mathrm{K}^*_p(\mathrm{X}^n) \qquad (\text{by } (3.5)).\end{aligned}$$

The products in $\mathrm{K}^*(\mathrm{X}^n)$ induce products in $\mathscr{K}^*(\mathrm{X})$, so that $\mathscr{K}^*(\mathrm{X})$ becomes a filtered ring. Also, for connected X, $\mathscr{K}^*(\mathrm{X})$ has an augmentation ε and a direct (group) decomposition $\mathscr{K}^*(\mathrm{X}) = \widetilde{\mathscr{K}}^*(\mathrm{X}) \oplus \mathbf{Z}$.

Lemma (4.1). — *Let* $f : \mathrm{Y} \to \mathrm{X}$ *be a continuous map,* X, Y *being* CW-*complexes. Let* ${}_n\mathrm{A} = \mathrm{K}^*(\mathrm{X}^n)$, ${}_n\mathrm{B} = \mathrm{K}^*(\mathrm{Y}^n)$, ${}_n\mathrm{A}' = \mathrm{Im}({}_{n+1}\mathrm{A} \to {}_n\mathrm{A})$, ${}_n\mathrm{B}' = \mathrm{Im}({}_{n+1}\mathrm{B} \to {}_n\mathrm{B})$. *Then* f *induces a homomorphism* ${}_nf^! : {}_n\mathrm{A}' \to {}_n\mathrm{B}'$ *which depends only on the homotopy class of* f.

Proof. — Let g, h be any two cellular maps homotopic to f. Then there exists a cellular homotopy Φ between g and h. Hence we have maps $g_n : \mathrm{Y}^n \to \mathrm{X}^n$, $h_n : \mathrm{Y}^n \to \mathrm{X}^n$ and $i_n \circ g_n \simeq i_n \circ h_n$ where $i_n : \mathrm{X}^n \to \mathrm{X}^{n+1}$ is the inclusion. This implies that $g^!_n \circ i^!_n = h^!_n \circ i^!_n$

[1] This restriction is not essential, but it covers the cases we are interested in.

these being homomorphisms ${}_{n+1}A \to {}_nB$. Thus g_n and h_n induce the same homomorphism ${}_nA' \to {}_nB'$ and this is the required ${}_nf^!$ depending only on the homotopy class of f.

Lemma **(4.2)**. — *In the notation of* (4.1) *suppose f is a homotopy equivalence. Then ${}_nf^!$ is an isomorphism.*

Proof. — This follows at once from (4.1).

Lemma **(4.3)**. — $\mathscr{K}^*(X)$ *is an invariant of homotopy type (as a filtered ring).*

Proof. — This follows from (4.2) and the fact that $\varprojlim {}_nA \cong \varprojlim {}_nA'$ (as filtered rings).

Note. — The above results apply equally to $\widetilde{\mathscr{K}}^*(X)$.

Lemma **(4.4)**. — *Let* X *and* Y *be connected* CW-*complexes of the same homotopy type, and put* ${}_nA = K^*(X^n)$, ${}_nB = K^*(Y^n)$, ${}_n\widetilde{A} = \widetilde{K}^*(X^n)$ ${}_n\widetilde{B} = \widetilde{K}^*(Y^n)$. *Then*

(i) $$\{{}_nA\} \text{ satisfies (ML)} \Leftrightarrow \{{}_nB\} \text{ satisfies (ML)},$$

(ii) $$\{{}_n\widetilde{A}\} \text{ satisfies (F)} \Leftrightarrow \{{}_n\widetilde{B}\} \text{ satisfies (F)}.$$

Proof. — This follows from (4.2) and the following facts:

$$\{{}_nA\} \text{ satisfies (ML)} \Leftrightarrow \{{}_nA'\} \text{ satisfies (ML)}$$
$$\{{}_n\widetilde{A}\} \text{ satisfies (F)} \Leftrightarrow \{{}_n\widetilde{A}'\} \text{ satisfies (F)},$$

where we adopt the notation of (4.1).

In view of (4.4) we may say that $\mathscr{K}^*(X)$ satisfies (ML) or that $\widetilde{\mathscr{K}}^*(X)$ satisfies (F), meaning that, for some cellular structure $\{K^*(X^n)\}$ satisfies (ML) or that $\{\widetilde{K}^*(X^n)\}$ satisfies (F).

Lemma **(4.5)**. — *Let* X *be a connected* CW-*complex, and let* $\{T^n\}$ *be an increasing sequence of finite connected sub-complexes of* X *with* $\bigcup_n T^n = X$. *Then.*

(i) $$\varprojlim K^*(T^n) \cong \mathscr{K}^*(X),$$

(ii) $$\{K^*(T^n)\} \text{ satisfies (ML)} \Leftrightarrow \mathscr{K}^*(X) \text{ satisfies (ML)},$$

(iii) $$\{\widetilde{K}^*(T^n)\} \text{ satisfies (F)} \Leftrightarrow \widetilde{\mathscr{K}}^*(X) \text{ satisfies (F)}.$$

Proof. — Let I be the directed set of all finite connected sub-complexes of X. Then the sets $\{X^n\}$, $\{T^n\}$ $(n \geq 1)$ (X^n being as before the n-skeleton) are confinal in I. The lemma now follows from (3.14).

Lemma **(4.6)**. — *Let* X *be a connected* CW-*complex with* $H^q(X, \mathbf{Z})$ *finite for all* $q > 0$. *Then* $\widetilde{\mathscr{K}}^*(X)$ *satisfies* (F).

Proof. — The hypotheses on X imply that $\operatorname{Im}\{H^q(X^{n+1}, \mathbf{Z}) \to H^q(X^n, \mathbf{Z})\}$ is finite for all $q > 0$. Hence, by (2.10), $\operatorname{Im}\{\widetilde{K}^*(X^{n+1}) \to \widetilde{K}^*(X^n)\}$ is finite, and so $\widetilde{\mathscr{K}}^*(X)$ satisfies (F).

Let G be a finite group. Then its classifying space B_G may be taken as a (connected) CW-complex (with finite skeletons) [10]. The homotopy type of B_G is

uniquely determined by G and hence, by (4.3), $\mathscr{K}^*(B_G)$ is a filtered ring depending only on G. Moreover, since an inner automorphism of G induces a map $B_G \to B_G$ homotopic to the identity on each finite skeleton [12, § 13.9], it follows that $\mathscr{K}^*(B_G)$ is invariant under inner automorphisms of G. For a direct definition of $\mathscr{K}^*(B_G)$ by means of universal operations on G-bundles, and for a direct proof of the invariance under inner automorphisms, see [1, § 4.6]. Since the cohomology groups $H^q(B_G, \mathbf{Z})$ are finite for $q>0$ we deduce, from (4.6).

Corollary **(4.7)**. — *Let* G *be a finite group. Then* $\widetilde{\mathscr{K}}^*(B_G)$ *satisfies* (F).

Suppose $f: Y \to X$ is a finite covering, X being a CW-complex. Then Y is a CW-complex and the n-skeletons Y^n of Y are simply the inverse images $f^{-1}(X^n)$ of the n-skeletons of X [13, § 4.5]. Thus f is the limit of maps $f_n : Y^n \to X^n$ of finite CW-complexes. It follows that the homomorphism $f_!$ of § 2 extends to the present infinite-dimensional complexes. Moreover the extended $f_!$ will have all the formal properties described in § 2. In particular (2.11) applies. Taking $X = B_G$ and Y the universal covering of X we deduce:

Proposition **(4.8)**. — $G\widetilde{\mathscr{K}}^*(B_G)$ *is annihilated by the order of* G.

Taking $X = B_G$ and $Y = B_H$, where H is a p-Sylow subgroup of G, we deduce:

Proposition **(4.9)**. — *The p-primary component of* $G\mathscr{K}^*(B_G)$ *is a direct factor of* $G\mathscr{K}^*(B_H)$, *where* H *is a p-Sylow subgroup of* G.

Remark. — Both these propositions are analogous to the corresponding results for cohomology.

Proposition **(4.10)**. — *For each prime p dividing the order of* G *let* G_p *be a p-Sylow subgroup of* G. *Then* $\mathscr{K}^*(B_G) \to \sum_p \mathscr{K}^*(B_{G_p})$ *is a monomorphism.*

Proof. — Suppose $x \in \operatorname{Ker}\{\mathscr{K}^*(B_G) \to \mathscr{K}^*(B_{G_p})\}$ for all p, $x \neq 0$. Since $\mathscr{K}^*(B_G)$ is a complete filtered group it is Hausdorff, and so $x \neq 0$ implies that there exists an integer n so that $x \in \mathscr{K}^*_n(B_G)$, $x \notin \mathscr{K}^*_{n+1}(B_G)$. But then x would define an element of $G^n\mathscr{K}^*(B_G)$ giving zero in each $G^n\mathscr{K}^*(B_{G_p})$. In view of (4.8) and (4.9) this is a contradiction, and so the proposition is proved.

The problem of generalizing the spectral sequences (2.4) and (2.6) to infinite-dimensional complexes presents serious difficulties (espectally (2.6)). We shall not attempt this problem in general but in the next section we deal with the case of classifying spaces of finite groups.

§ 5. The spectral sequence of a normal subgroup.

Let G be a finite group, V a normal subgroup and put $S = G/V$. Let B_G, B_S be the classifying spaces of G, S, and let $E_G \to B_G$, $E_S \to B_S$ be the universal bundles (i.e. the universal coverings). Then we have a factorization (cf. [9, Satz 3.44])

$$E_G \to B_V \to B_G$$

where $B_V = E_G/V$ is a classifying space for V, and $B_V \to B_G$ is the bundle associated to the universal bundle with fibre S. Since V is normal $B_V \to B_G$ is moreover a principal S-bundle.

An element $g \in G$ defines a map $E_G \to E_G$ given by $x \to xg$. Since V is normal we have $xgV = xVg$ and so g induces a map $\theta_g : B_V \to B_V$. On the other hand g induces an automorphism $v \to g^{-1}vg$ of V and hence a weak [1] homotopy class of maps $B_V \to B_V$. It is easy to check from the definitions that θ_g is a representative map of this class. Moreover θ_g depends only on the coset $s = gV$ and it describes the way in which s operates on the principal S-bundle B_V.

For simplicity put $X = B_G$, $Y = B_S$, $A = B_V$, $B = E_S$. Let X^n, Y^n be the n-skeletons of X, Y and let A^n, B^n be their inverse images in A, B. As already observed in § 4, A^n, B^n are the n-skeletons of a CW-structure on A, B. Moreover these CW-structures are invariant under the operation of S [13, § 4.5]. Hence $Z = (A \times B)/S$ will have an induced CW-structure and $Z^n = (A^n \times B^n)/S$ will be a finite sub-complex, connected for $n \geq 1$. Now we have two fibrations

$$\begin{tikzcd} & Z \arrow[dl, "B"'] \arrow[dr, "A"] & \\ X & & Y \end{tikzcd}$$

which are the limits, under inclusion, of the fibrations

$$\begin{tikzcd} & Z^n \arrow[dl, "B^n"'] \arrow[dr, "A^n"] & \\ X^n & & Y^n \end{tikzcd}$$

Since $B = E_S$ is contractible it follows that $Z \to X$ is a homotopy equivalence. Hence, by (4.3), we have $\mathscr{K}^*(Z) \cong \mathscr{K}^*(B_G)$. Since $Z = \bigcup_n Z^n$ it follows from (4.4), (4.5) and (4.7) that $\{\widetilde{\mathscr{K}}^*(Z^n)\}$ satisfies (F).

From the fibrations $Z^n \to Y^n$ we obtain, by (2.6), a spectral sequence $\{{}_nE_r\}$ with

$${}_nE_2^p = H^p(Y^n, \mathbf{K}^*(\mathbf{A}^n))$$
$${}_nE_\infty^p = K_p^*(Z^n)/K_{p+1}^*(Z^n).$$

Here $\mathbf{K}^*(\mathbf{A}^n)$ is the local coefficient system associated to the operation of

$$\pi_1(Y^n) \cong \pi_1(Y) \cong S \quad (n \geq 2)$$

on A^n defined by the fibration $A^n \to X^n$, and [2] $K_p^*(Z^n)$ is the filtration on $K^*(Z^n)$ induced from Y^n. By (2.8) we can decompose ${}_nE_r = {}_n\widetilde{E}_r \oplus \mathbf{Z}$ where $\mathbf{Z}$ is the trivial spectral sequence, ${}_n\widetilde{E}_r^p = {}_nE_r^p$ for $p > 0$ and

$${}_n\widetilde{E}_2^0 = H^0(Y^n, \widetilde{\mathbf{K}}^*(\mathbf{A}^n)),$$
$${}_n\widetilde{E}_\infty^0 = \widetilde{K}_0^*(Z^n)/\widetilde{K}_1^*(Z^n).$$

(1) I.e. defined on each skeleton.
(2) The suffix of (2.6) is omitted here to simplify notation.

Since Z^n is a finite CW-complex there is no convergence problem for $\{{}_n E_r\}$ as $r \to \infty$ (n fixed). Hence $\{{}_n\widetilde{E}_r, \widetilde{K}^*(Z^n)\}$ is an inverse system of strongly convergent spectral sequences in the sense of § 3. We wish to apply (3.19). Now condition *b)* of (3.19) has already been verified so it remains to consider *a)*.

Since A^n is a finite CW-complex, $K^*(A^n)$ is a finitely-generated group (2.9). Hence ${}_n\widetilde{E}_2^p$ is finitely-generated. But, for $n > p$, we have:

$${}_n\widetilde{E}_2^p \cong H^p(Y, \mathbf{K}^*(\mathbf{A}^n)) \cong H^p(S, K^*(A^n)),$$

since $Y = B_S$. Hence ${}_n\widetilde{E}_2^p$ for $n > p > 0$, is annihilated by the order of S and so is finite. Thus, for $p > 0$, $\{{}_n\widetilde{E}_2^p\}$ satisfies (F). For $p = 0$ and $n > 0$ we have:

$${}_n\widetilde{E}_2^0 = H^0(S, \widetilde{K}^*(A^n)) \cong \widetilde{K}^*(A^n)^S \text{ (the invariants).}$$

By (4.7) we know that $\{\widetilde{K}^*(A^n)\}$ satisfies (F) (since $A = B_V$) and so, by (3.12), $\{\widetilde{K}^*(A_n)^S\}$ also satisfies (F).

Conditions *a)* and *b)* of (3.19) therefore hold, and so we obtain a strongly convergent spectral sequence $\{\varprojlim_n \widetilde{E}_r, \widetilde{\mathscr{K}}^*(Z)\}$. Adding the trivial spectral sequence $\mathbf{Z}$ does not affect the convergence and gives $\varprojlim H^p(S, K^*(A^n)) \Rightarrow \mathscr{K}^*(Z)$.

Now

$$H^p(S, K^*(A^n)) = H^p(\mathrm{Hom}_\Lambda(L_*, K^*(A^n)))$$

where $\Lambda = \mathbf{Z}[S]$ and L_* is the standard Λ-free resolution of $\mathbf{Z}$ [7, chapter X]. Since $\{\widetilde{K}^*(A^n)\}$ satisfies (F), $\{K^*(A^n)\}$ satisfies (ML) and so for each p $\{\mathrm{Hom}_\Lambda(L_p, K^*(A^n))\}$ satisfies (ML). Moreover as already observed $\{H^p(S, K^*(A^n))\}$ satisfies (F) for $p > 0$, and $\{H^0(S, \widetilde{K}^*(A^n))\}$ satisfies (F). Hence for all p $\{H^p(S, K^*(A^n))\}$ satisfies (ML) (adding $\mathbf{Z}$ for $p = 0$). Hence (1), by (3.9),

$$\varprojlim H^p(S, K^*(A^n)) \cong H^p(S, \mathscr{K}^*(A)).$$

Since $\mathscr{K}^*(A) = \mathscr{K}^*(B_V)$ and $\mathscr{K}^*(Z) \cong \mathscr{K}^*(B_G)$ we have established the following theorem.

Theorem **(5.1).** — *Let* G *be a finite group,* V *a normal subgroup and* $S = G/V$. *Then* $\mathscr{K}^*(B_G)$ *has a filtration defined relative to* S *(denoted by a subscript* S*), and we have a strongly convergent spectral sequence*: $\quad H^*(S, \mathscr{K}^*(B_V)) \Rightarrow \mathscr{K}^*(B_G)_S$.

Either by taking $S = G$ in (5.1) or more directly by repeating the proof and using (2.4) instead of (2.6) we obtain

Theorem **(5.2).** — *Let* G *be a finite group. Then there is a strongly convergent spectral sequence*:

$$H^*(G, \mathbf{Z}) \Rightarrow \mathscr{K}^*(B_G).$$

All the properties of the spectral sequences (2.6) and (2.4) go over to (5.1) and (5.2). In particular this applies to the product structures and to the conditions under which $d_{2r} = 0$.

(1) This could have been incorporated in the proof of (3.19) by starting the spectral sequence with E_1.

In (5.1) $\mathscr{H}^*(B_V)$ is an S-module, the operation being induced by conjugation as explained earlier (see [1, § 4.6] for a direct definition of this operation).

The filtrations of (5.1) and (5.2) are such that $\mathscr{H}^*_p(B_G)_S \subset \mathscr{H}^*_p(B_G)$. Thus the "S-topology" of $\mathscr{H}^*(B_G)$ is *finer* than the "G-topology".

A notable case of (5.2) is the following:

Corollary **(5.3)**. — *If* $H^q(G, \mathbf{Z}) = 0$ *for all odd* q, *then* $H^*(G, \mathbf{Z}) \cong G\mathscr{H}^*(B_G)$ *(as graded rings).*

Proof. — The hypothesis and *d)* of (2.4) imply that $d_r = 0$ for all r. The corollary then follows from (3.18).

A similar result holds for (5.1).

Corollary **(5.4)**. — *If in* (5.1) $H^q(S, \mathscr{H}^*(B_V)) = 0$ *for all odd* q, *and* $\mathscr{H}^1(B_V) = 0$, *then* $H^*(S, \mathscr{H}^*(B_V)) \cong G\mathscr{H}^*(B_G)_S$ *(as graded rings).*

Proof. — This follows from *c)* of (2.6) and (3.18).

§ 6. The representation ring R(G).

Let G be a finite group. We denote by R(G) the free abelian group generated by the equivalence classes of irreducible complex (or unitary) representations of G. Thus, if $\xi_1, \ldots, \xi_n$ are the (classes of) irreducible representations of G, every element of R(G) can be written uniquely as

$$\rho = \sum_{i=1}^{n} r_i \xi_i, \qquad r_i \in \mathbf{Z}.$$

The (classes of) representations of G correspond to the "positive" elements of R(G), i.e. those with $r_i \geqslant 0$ for all i (but not all $r_i = 0$).

The tensor product makes R(G) into a ring. We shall call this the *representation ring* of G — it is isomorphic to the *character ring* of G. In this section we shall identity these two rings.

We define an augmentation $\varepsilon : R(G) \to Z$ by $\varepsilon(\xi_i) = \dim \xi_i$, and we denote by I(G) the kernel of ε. We shall consider R(G) with the I(G)-adic (or augmentation) topology, and its completion $\widehat{R(G)}$ in this topology. The main result of this section will be

Theorem **(6.1)**. — *Let* H *be a subgroup of* G. *Then the* I(H)*-adic topology of* R(H) *is the same as its* I(G)*-adic topology* (R(H) *being viewed as* R(G)*-module via the restriction homomorphism* $R(G) \to R(H)$).

Let G have order g. Let $\chi_1, \ldots, \chi_n$ be the characters of $\xi_1, \ldots, \xi_n$. Then

$$R(G) = \sum_i \mathbf{Z}\chi_i.$$

Let $\theta = \exp(2\pi i/h)$ where h is a multiple of g, and put $A = \mathbf{Z}[\theta]$, so that

$$A = \mathbf{Z} + \mathbf{Z}\theta + \ldots + \mathbf{Z}.\theta^{\varphi(h)-1} \qquad (\varphi \text{ the Euler function}).$$

Define
$$R_A(G) = R(G) \otimes_{\mathbf{Z}} A = \sum_i A\chi_i.$$

Now a character is a (class) function on G and its values are in A, being sums of eigenvalues θ^ν of unitary representations of period dividing g. Hence we may regard $R_A(G)$ as a subring of the ring [1] A^G of *all* A-valued functions on G.

Lemma **(6.2)**. — (i) *Every prime ideal of* $R_A(G)$ *is the restriction of a prime ideal of* A^G.
(ii) *Every prime ideal of* $R(G)$ *is the restriction of a prime ideal of* $R_A(G)$.

Proof. — A^G is a finitely-generated $\mathbf{Z}$-module. In fact if, for each $S \in G$, we let e_S denote the function taking the value 1 on S and 0 elsewhere, then

$$A^G = \sum_{S \in G} \sum_{\nu=0}^{\varphi(h)-1} \mathbf{Z}\theta^\nu e_S.$$

$\mathbf{Z}$ is imbedded in $R(G)$ by the trivial representation, in A^G by the constant integer valued functions. Since we have inclusions

$$\mathbf{Z} \subset R(G) \subset R_A(G) \subset A^G,$$

it follows that

a) A^G is a finitely-generated $R_A(G)$-module;

b) $R_A(G)$ is a finitely-generated $R(G)$-module.

By the theorem of Cohen-Seidenberg [14, p. 257, Th. 3] *a)* and *b)* imply (i) and (ii) respectively.

We may remark, at this point, that all the rings occurring here are finitely-generated $\mathbf{Z}$-modules and so certainly Noetherian.

The prime ideals in A^G are easy to describe, because A^G is just a sum of g copies of A. If $S \in G$ and $\mathfrak{p}$ is a prime ideal of A, then the set of functions $\psi \in A^G$ such that $\psi(S) \in \mathfrak{p}$ is a prime ideal of A^G, and every prime ideal of A^G is of this type. We denote the restriction of this prime ideal to $R_A(G)$ by $P_{\mathfrak{p},S}$. Thus

$$P_{\mathfrak{p},S} = \{\chi \in R_A(G) \mid \chi(S) \in \mathfrak{p}\}.$$

By (6.2) (i) we know that every prime ideal of $R_A(G)$ is of this form for some $\mathfrak{p}$ and some S.

If $\mathfrak{p} \neq (0)$ then $\mathfrak{p} \cap \mathbf{Z} = p\mathbf{Z}$ for some prime $p \neq 0$ of $\mathbf{Z}$, $\mathfrak{p}$ is a maximal ideal of A and $A/\mathfrak{p}$ is a finite field of characteristic p. We then define $S_\mathfrak{p}$ by the decomposition $S = S_\mathfrak{p}.B$, where $S_\mathfrak{p}$ and B are powers of S, $S_\mathfrak{p}$ has order *prime to p* and B has order a power of p. If $\mathfrak{p} = (0)$ we define $S_\mathfrak{p} = S$. $S_\mathfrak{p}$ is called the $\mathfrak{p}$-regular factor of S.

Lemma **(6.3)**. — $P_{\mathfrak{p},S} \supset P_{\mathfrak{p}',S'}$ *if and only if* (i) $\mathfrak{p} \supset \mathfrak{p}'$ *and* (ii) $S_\mathfrak{p}$ *and* $S'_\mathfrak{p}$ *are conjugate in* G.

Proof. — Suppose first that (i) and (ii) hold. To prove that $P_{\mathfrak{p},S} \supset P_{\mathfrak{p}',S'}$ it will be sufficient to show that, for any $\mathfrak{p}$, S and $\chi \in R_A(G)$ we have

$$\chi(S) \equiv \chi(S_\mathfrak{p}) \bmod \mathfrak{p}.$$

This is trivial if $\mathfrak{p} = (0)$, so we may suppose $\mathfrak{p} \cap \mathbf{Z} = p\mathbf{Z}$, $p \neq 0$. Restricting χ to the cyclic subgroup generated by S we see that it is sufficient to deal with the case where G

[1] Elsewhere this notation is used for the invariants, but there should be no confusion.

is cyclic and generated by S. Also we may suppose χ is irreducible, hence one-dimensional, hence multiplicative: $\chi(S_1S_2)=\chi(S_1).\chi(S_2)$.

Let $f_p: A \to A/\mathfrak{p}$ be the canonical homomorphism. Then from $S=S_p.B$ we obtain

$$f_p(\chi(S))=f_p(\chi(S_p)).f_p(\chi(B)).$$

Since $B^{p^\nu}=1$ for some ν, we have $(f_p(\chi(B)))^{p^\nu}=1$ for some ν. But $A/\mathfrak{p}$ is a finite field of characteristic p, hence $f_p(\chi(B))=1$, and this completes the proof.

Conversely suppose $P_{\mathfrak{p},S}\supset P_{\mathfrak{p}',S'}$. Then

$$\mathfrak{p}=P_{\mathfrak{p},S}\cap A\supset P_{\mathfrak{p}',S'}\cap A=\mathfrak{p}'.$$

Suppose first that $\mathfrak{p}\neq(0)$. Then according to (1) [5, Lemma 3] there exists $\eta\in R_A(G)$ such that

(i) η has values in $\mathbf{Z}$;

(ii) $\eta(T)=0$ if T_p is not conjugate to S_p;

(iii) $\eta(T)\equiv 1 \pmod p$ if T_p is conjugate to S_p.

If S'_p and S_p were not conjugate we would have $\eta\in P_{\mathfrak{p}',S'}$ but $\eta\notin P_{\mathfrak{p},S}$. This gives a contradiction and so S_p and S'_p are conjugate. If $\mathfrak{p}=(0)$ and S and S′ are not conjugate, then as is well-known there exists $\eta\in R(G)$ with $\eta(S')=0$, $\eta(S)\neq 0$, which gives a contradiction also in this case.

This lemma leads at once to the following description of the "scheme" of the prime ideals of $R_A(G)$.

Proposition **(6.4)**. — *The prime ideals of* $R_A(G)$ *are all of the form* $P_{\mathfrak{p},S}$. *Two such ideals* $P_{\mathfrak{p},S}$ *and* $P_{\mathfrak{p}',S'}$ *coincide if and only if* $\mathfrak{p}=\mathfrak{p}'$ *and* S_p *and* S'_p *are conjugate in* G. *If* $\mathfrak{p}=(0)$, $P_{\mathfrak{p},S}$ *is a minimal prime ideal, while if* $\mathfrak{p}\neq(0)$ $P_{\mathfrak{p},S}$ *is a maximal prime ideal. The maximal prime ideals containing* $P_{0,S}$ *are the ideals* $P_{\mathfrak{p},S}$ *with* $\mathfrak{p}\neq(0)$. *The minimal prime ideals contained in* $P_{\mathfrak{p},S}(\mathfrak{p}\neq(0))$ *are the* $P_{0,S'}$ *with* S'_p *conjugate to* S_p.

Lemma (2) **(6.5)**. — $P_{0,1}=A.I(G)$.

Proof. — Trivially $A.I(G)\subset P_{0,1}$. On the other hand let $\chi\in P_{0,1}$ and write

$$\chi=\sum_{\nu=0}^{\varphi(h)-1}\chi_\nu\theta^\nu$$

with $\chi_\nu\in R(G)$. Then $\chi(1)=\sum_\nu\chi_\nu(1).\theta^\nu=0$, with $\chi_\nu(1)\in\mathbf{Z}$. This implies $\chi_\nu(1)=0$ for all ν and $\chi_\nu\in I(G)$, i.e. $\chi\in A.I(G)$.

Now let H be a subgroup of G. To distinguish we shall write $P_{\mathfrak{p},S}(G)$ instead of $P_{\mathfrak{p},S}$. Let $\rho: R_A(G)\to R_A(H)$ be the restriction homomorphism.

Lemma **(6.6)**. — *Suppose* $\rho^{-1}(P_{\mathfrak{p},S}(H))=P_{\mathfrak{p}',1}(G)$. *Then* $P_{\mathfrak{p},S}(H)=P_{\mathfrak{p}',1}(H)$.

Proof. — Since $S\in H$, $\rho^{-1}(P_{\mathfrak{p},S}(H))=P_{\mathfrak{p},S}(G)$. Hence $P_{\mathfrak{p},S}(G)=P_{\mathfrak{p}',1}(G)$, and so by (6.4) $\mathfrak{p}=\mathfrak{p}'$ and S_p is conjugate in G to $1_p=1$. Hence $S_p=1$ and so S_p is

(1) This lemma is the main step in the proof of Brauer's theorem given in [5]. Brauer's theorem itself will be needed in § 11.

(2) We denote the identity of G by 1.

conjugate to 1_p in H. Hence by (6.4) applied to H we conclude that $P_{p,S}(H)=P_{p',1}(H)$ as required.

Lemma **(6.7)**. — *The prime ideals of* R(H) *which contain* $\rho I(G)$ *are the same as those which contain* I(H).

Proof. — Trivially $\rho I(G)\subset I(H)$. Hence we must show that, if P_0 is a prime ideal of R(H) which contains $\rho I(G)$, then $P_0\supset I(H)$. By (6.2) (ii) applied to H there exists $\mathfrak{p}$, S so that $P_0=R(H)\cap P_{\mathfrak{p},S}(H)$. Then $\rho I(G)\subset P_0$ implies $\rho I(G)\subset P_{\mathfrak{p},S}(H)$. Hence by (6.5)

$$\rho P_{0,1}(G)=\rho(A.I(G))\subset A.\rho I(G)\subset P_{\mathfrak{p},S}(H)$$

and so

$$P_{0,1}(G)\subset\rho^{-1}(P_{\mathfrak{p},S}(H)).$$

By (6.4) this implies $\rho^{-1}(P_{\mathfrak{p},S}(H))=P_{\mathfrak{p}',1}(G)$ and so by (6.6) $P_{\mathfrak{p},S}(H)=P_{\mathfrak{p}',1}(H)$. Hence $I(H)\subset P_{\mathfrak{p},S}(H)$, and therefore $I(H)\subset R(H)\cap P_{\mathfrak{p},S}(H)=P_0$. Q.E.D.

Theorem (6.1) follows at once from (6.7) and the fact that, in a Noetherian ring, the $\mathfrak{a}$-adic topology is the same as the $\mathfrak{a}'$-adic topology where $\mathfrak{a}'$ is the radical of $\mathfrak{a}$ (i.e. the intersection of all the prime ideals containing $\mathfrak{a}$). For the proof of this statement see [11, p. 14, Th. 1 and p. 22, Prop. 8].

Lemma **(6.8)**. — *Let* B *be a Noetherian ring with no (non-zero) nilpotent elements. Let* $\mathfrak{b}$ *be a prime ideal of* B, *and let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ *be the minimal prime ideals of* B *numbered so that*

$$\text{for } 1\leqslant i\leqslant m \qquad \mathfrak{b}+\mathfrak{p}_i\neq B \qquad \text{and}$$
$$\text{for } i>m \qquad \mathfrak{b}+\mathfrak{p}_i=B.$$

Then

$$\bigcap_{n=1}^{\infty}\mathfrak{b}^n=\mathfrak{p}_1\cap\mathfrak{p}_2\cap\ldots\cap\mathfrak{p}_m.$$

Proof. — In any Noetherian ring we have [11, p. 14, Th. 1]

$$\mathfrak{N}=\bigcap_{i=1}^{k}\mathfrak{p}_i,$$

where $\mathfrak{N}$ is the ideal of nilpotent elements and the $\mathfrak{p}_i$ are the minimal prime ideals. With the hypothesis of the lemma we have $\mathfrak{N}=(0)$. Thus $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are the primary components of (0) and the lemma now follows from [14, p. 218 Corollary].

We observe that the condition $\mathfrak{b}+\mathfrak{p}_i\neq B$ in (6.8) is equivalent to: there exists a maximal prime ideal $\mathfrak{q}$ with $\mathfrak{b}+\mathfrak{p}_i\subset\mathfrak{q}$. If $\mathfrak{b}+\mathfrak{p}_i\neq B$ we take $\mathfrak{q}$ to be a maximal ideal containing $\mathfrak{b}+\mathfrak{p}_i$ and recall that a maximal ideal is necessarily prime.

We now apply (6.8) with $R_A(G)$ for B and $P_{0,1}$ for $\mathfrak{b}$. From (6.4) it follows that the maximal prime ideals containing $P_{0,1}$ are the $P_{\mathfrak{p},1}$ $(\mathfrak{p}\neq(0))$, and that the minimal prime ideals contained in $P_{\mathfrak{p},1}$ are the $P_{0,S'}$ with S'_p conjugate to $1_p=1$, i.e. S' of order a power of p. Hence (6.8), together with (6.5), gives

Lemma **(6.9)**. — $\bigcap_{n=1}^{\infty}(A.I(G))^n=\{\chi\in R_A(G)\,|\,\chi(S)=0$ *for all* $S\in G$ *having prime power order*$\}$.

Suppose now that J is any ideal of R(G). Then A.J is the ideal of $R_A(G)$ consisting of elements of the form

$$\sum_{\nu=0}^{\varphi(h)-1} j_\nu \theta^\nu, \qquad j_\nu \in J.$$

Since the θ^ν form a *free* basis for $R_A(G)$ over R(G) it follows that $A.J \cap R(G) = J$. Taking $J = I(G)^n$, and observing that $A.I(G)^n = (A.I(G))^n$, we deduce from (6.9):

Proposition (**6.10**). — $\bigcap_{n=1}^{\infty} I(G)^n = \{\chi \in R(G) \mid \chi(S) = 0$ *for all* $S \in G$ *having prime power order*$\}$.

(6.10) and (3.2) together imply

Proposition (**6.11**). — *If* G *is a p-group then* $R(G) \to \widehat{R(G)}$ *is a monomorphism.*

In general, for each prime p dividing the order of G, let G_p be a p-Sylow subgroup of G. Then we have a restriction homomorphism

$$R(G) \to \sum_p R(G_p).$$

Proposition (**6.12**). — *The kernels of the two homomorphisms*

$$R(G) \to \widehat{R(G)} \quad \text{and} \quad R(G) \to \sum_p R(G_p)$$

coincide.

Proof. — Denote the homomorphisms by α, β respectively. In (6.10) Ker α is explicitly determined and from this it is immediate that $\operatorname{Ker}\alpha \subset \operatorname{Ker}\beta$. Conversely suppose $\chi \in R(G)$ is in Ker β. Then $\chi(S) = 0$ for $S \in G_p$. But every element of G of order p is conjugate to an element S of G_p. Hence $\chi(S) = 0$ for all S of prime power order, i.e. $\chi \in \operatorname{Ker}\alpha$.

Next we shall examine the quotients $I(G)^n / I(G)^{n+1}$.

Proposition (**6.13**). — *Let g be the order of* G, *and let* $n > 0$. *Then* $I(G)^n/I(G)^{n+1}$ *is a finite group annihilated by g.*

Proof. — Since R(G) is a finitely-generated group the same is true of $I(G)^n/I(G)^{n+1}$. Hence it will be sufficient to show that $g.I(G)^n \subset I(G)^{n+1}$ for $n > 0$.

For any subgroup H of G we have the two homomorphisms:

$$i^* : R(G) \to R(H) \qquad \text{(restriction)}$$
$$i_* : R(H) \to R(G) \qquad \text{(induced representation)},$$

and the formula ((*e*) of § 1)

$$i_*(i^*(\alpha).\beta) = \alpha . i_*(\beta) \qquad \alpha \in R(G),\ \beta \in R(H).$$

In particular we may take $H = 1$, and apply the formula with $\alpha \in I(G)^n$ ($n > 0$) and $\beta = 1$. Then $i^*(\alpha) = 0$ and so we deduce $\alpha . i_*(1) = 0$. Now $i_*(1)$ has augmentation g (in fact $i_*(1)$ is the regular representation of G), and so $g - i_*(1) \in I(G)$. Hence

$$g\alpha = (g - i_*(1)).\alpha \in I(G)^{n+1},$$

which completes the proof.

Remark. — This proof is formally similar to that of (4.8) or to the corresponding result for cohomology.

§ 7. Statement of the main theorems.

Let G be a finite group, ξ a principal G-bundle over a finite CW-complex X. Then, as observed in § 1, each complex representation ρ of G defines a complex vector bundle $\rho(\xi)$ over X. Because of properties (1)-(5) of § 1 this extends to a ring homomorphism $R(G) \to K^0(X)$, compatible with inverse images, direct images and exterior powers. If X is an infinite dimensional CW-complex with finite skeletons X^n, then the homomorphisms $R(G) \to K^0(X^n)$ are compatible with each other and so define a homomorphism $R(G) \to \mathscr{K}^0(X) = \varprojlim K^0(X^n)$. In particular, taking $X = B_G$ and ξ the universal G-bundle we obtain a homomorphism

$$\alpha : R(G) \to \mathscr{K}^0(B_G).$$

Following α by the inclusion of $\mathscr{K}^0(B_G)$ in $\mathscr{K}^*(B_G)$ we obtain a homomorphism, which we still denote by α

$$\alpha : R(G) \to \mathscr{K}^*(B_G).$$

α is a ring homomorphism and commutes with inverse images, direct images (for $H \subset G$), exterior powers and augmentation (cf. § 4 and § 6).

If $\rho \in I(G)$ then $\alpha(\rho) \in \widetilde{\mathscr{K}}^*(B_G) = \mathscr{K}^*_2(B_G)$ (by (2.2)), and so

(7.1) $$\alpha(I(G)^n) \subset \mathscr{K}^*_{2n}(B_G).$$

Thus α is *continuous*, $R(G)$ having the $I(G)$-adic topology, and $\mathscr{K}^*(B_G)$ having the filtration (or inverse limit) topology. Hence α induces a homomorphism $\hat{\alpha}$ of the completions. Since $\mathscr{K}^*(B_G)$ is an inverse limit and hence complete (§ 3), it follows that $\hat{\alpha}$ is a homomorphism:

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G).$$

Our main theorem is then:

Theorem **(7.2)**. — $\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$ *is a topological isomorphism.*

Obvious corollaries are:

Corollary **(7.3)**. — $\mathscr{K}^1(B_G) = 0$.

Corollary **(7.4)**. — $\mathscr{K}^0(B_G)$ *has no elements of finite order.*

Corollary **(7.5)**. — *The topology on* $R(G)$ *induced by* α *from the filtration on* $\mathscr{K}^*(B_G)$ *coincides with the* $I(G)$*-adic topology.*

Combining (7.2) with (5.2) we obtain

Theorem **(7.6)**. — *Let* G *be a finite group, then* $\widehat{R(G)}$ *has a filtration for which there is a strongly convergent spectral sequence*

$$H^*(G, \mathbf{Z}) \Rightarrow \widehat{R(G)}.$$

This is the spectral sequence referred to in the introduction, bearing in mind that $GR(G) \cong GR(\widehat{R(G)})$ (3.10). Properties *a)-d)* of the introduction follow from *a)-d)* of (2.4). Property *e)* follows from (2.2).

From (2.5), and the fact that the mapping $\rho \to \rho(\xi)$ commutes with exterior

powers ((3) of § 1) and so in particular with the determinant operation "det", we deduce:

Proposition **(7.7)**. — *The 2-dimensional part of* (7.6) *gives an isomorphism*

$$R_2(G)/R_4(G) \cong H^2(G, \mathbf{Z}).$$

This isomorphism is induced by the mapping $\rho - \varepsilon(\rho) \to \det \rho$, *followed by the isomorphism* $\mathrm{Hom}(G, \mathbf{C}^*) \cong H^2(G, \mathbf{Z})$ *of* (1.3).

Combining (7.2) with (5.1) we obtain the following generalization of (7.6):

Theorem **(7.8)**. — *Let* G *be a finite group,* V *a normal subgroup,* $S = G/V$. *Then* R(G) *has a filtration defined relative to* S *(denoted by a subscript* S*) for which there is a strongly convergent spectral sequence*:

$$H^*(S, \widehat{R(V)}) \Rightarrow \widehat{R(G)}_S.$$

Here $\widehat{R(V)}$ *is an* S-*module, the operation of* S *being induced by conjugation in* G. *This spectral sequence is contravariant in* (V, G, S), *has products and is such that all* $d_{2r} = 0$.

The extra properties of the spectral sequence in (7.8) follow as before from (2.6).

Lemma **(7.9)**. — *To prove* (7.2) *it is sufficient to prove*

(i) $\hat{\alpha}$ *is a monomorphism*;

(ii) $\alpha R(G)$ *is dense in* $\mathscr{K}^*(B_G)$.

Proof. — We decompose

$$R(G) = \mathbf{Z} \oplus I(G), \qquad \mathscr{K}^*(B_G) = \mathbf{Z} \oplus \widetilde{\mathscr{K}}^*(B_G).$$

Then α and $\hat{\alpha}$ decompose accordingly. Consider then

1) $$\hat{\alpha} : \widehat{I(G)} \to \widetilde{\mathscr{K}}^*(B_G).$$

Now $\widehat{I(G)}$ is an inverse limit of finite groups (6.13) and so is a compact Hausdorff group. The same applies to $\widetilde{\mathscr{K}}^*(B_G)$, by (4.7) and (3.13). Hence $\hat{\alpha}(\widehat{I(G)})$ is *closed* in $\widetilde{\mathscr{K}}^*(B_G)$. Now (ii) implies that $\hat{\alpha}(\widehat{I(G)})$ is dense in $\widetilde{\mathscr{K}}^*(B_G)$ and so 1) must be an epimorphism. Together with (i) this proves 1) is an isomorphism. Since $\widehat{I(G)}$ and $\widetilde{\mathscr{K}}^*(B_G)$ are compact Hausdorff groups any continuous isomorphism between them must be a homeomorphism. The same is then true for $\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$.

§ 8. Cyclic Groups.

In this section we shall prove (7.2) for cyclic groups and then derive (i) of (7.9) for general finite groups.

Let G be a cyclic group of order n, and let ρ the representation which maps a generator of G to $\exp \frac{2\pi i}{n}$. Then from (1.3) it follows that ρ corresponds to a generator x of $H^2(G, \mathbf{Z})$. Now it is well-known [7, p. 251] that

1) $H^*(G, \mathbf{Z}) = \mathbf{Z}[x]$, where $nx = 0$, $\mathbf{Z}[x]$ denoting the graded ring of polynomials

in x. In particular there is no odd-dimensional cohomology and so, from (5.3) we have an isomorphism

2) $$H^*(G, \mathbf{Z}) \cong G\mathcal{K}^*(B_G).$$

Now $R(G) = \mathbf{Z}[\rho]$ where $\rho^n = 1$. If we put $\sigma = \rho - 1$, then $R(G) = \mathbf{Z}[\sigma]$ where $(1+\sigma)^n = 1$, and $I(G)$ is the ideal (σ). Since

$$0 = (1+\sigma)^n - 1 \equiv n\sigma \bmod \sigma^2,$$

it follows that, for $k > 0$, $I(G)^k/I(G)^{k+1}$ is cyclic of order n generated by the class of σ^k. If we filter $R(G)$ by defining $R_{2k}(G) = R_{2k-1}(G) = I(G)^k$ it follows that we have

3) $GR(G) = \mathbf{Z}[\bar{\sigma}]$ where $n\bar{\sigma} = 0$, and $\bar{\sigma}$ is the residue class of σ mod $I(G)^2$.

Consider now the homomorphism

$$\alpha : R(G) \to \mathcal{K}^*(B_G).$$

It is a homomorphism of filtered rings, and so induces a homomorphism of graded rings:

$$G\alpha : GR(G) \to G\mathcal{K}^*(B_G).$$

If we identify $G\mathcal{K}^*(B_G)$ with $H^*(G, \mathbf{Z})$ by 2) we find, from (2.5), that

$$G\alpha(\bar{\sigma}) = c_1\alpha(\rho) = x.$$

Hence from 1) and 3) $G\alpha$ is an isomorphism. From (3.10) we deduce:

Proposition **(8.1)**. — *Let* G *be a cyclic group and filter* R(G) *by putting*

$$R_{2k-1}(G) = R_{2k}(G) = I(G)^k.$$

Then $\widehat{R(G)}$ *has an induced filtration, and* $\hat{\alpha} : \widehat{R(G)} \to \mathcal{K}^*(B_G)$ *is an isomorphism of filtered groups.*

This is, for cyclic groups, a more precise result than (7.2).

We proceed now to prove (i) of (7.9).

Lemma **(8.2)**. — *Let* G *be a finite group,* $\{G_\lambda\}$ *the family of all cyclic subgroups of* G. *Then* $R(G) \to \sum_\lambda R(G_\lambda)$ *(given by the restriction) is a monomorphism.*

Proof. — If $\rho \in R(G)$ gives zero in each G_λ, then $\chi_\rho | G_\lambda = 0$, where χ_ρ is the character of ρ. Since $G = \bigcup_\lambda G_\lambda$, this implies $\chi_\rho = 0$ and so $\chi = 0$.

Lemma **(8.3)**. — *With the same notation as* (8.2)

$$\widehat{R(G)} \to \sum_\lambda R(G_\lambda)^\wedge$$

is a monomorphism (where each completion is with respect to the augmentation ideal of the corresponding group).

Proof. — By (8.2) we have an exact sequence

$$0 \to R(G) \to \sum_\lambda R(G_\lambda).$$

By (6.1) the $I(G_\lambda)$-topology of $R(G_\lambda)$ is the same as the $I(G)$-topology. Hence regarding $R(G)$ and $\sum_\lambda R(G_\lambda)$ as $R(G)$-modules, and completing with respect to the $I(G)$-topology, we get (by (3.16)) an exact sequence:

$$0 \to R(G) \to (\sum_\lambda R(G_\lambda))^\wedge.$$

But $(\sum_\lambda R(G_\lambda))^\wedge \cong \sum_\lambda R(G_\lambda)^\wedge$, and so the lemma is proved.

Proposition **(8.4)**. — *For any finite group* G

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$$

is a monomorphism.

Proof. — Let $\{G_\lambda\}$ be the family of all cyclic subgroups of G. Then we have a commutative diagram

$$\begin{tikzcd} \widehat{R(G)} \arrow[r, "\theta"] \arrow[d, "\hat{\alpha}"'] & \sum_\lambda R(G_\lambda)^\wedge \arrow[d, "\varphi"] \\ \mathscr{K}^*(B_G) \arrow[r] & \sum_\lambda \mathscr{K}^*(B_{G_\lambda}). \end{tikzcd}$$

Ker $\theta = 0$ (8.3) and φ is an isomorphism (8.1). Hence Ker $\hat{\alpha} = 0$ as required.

§ 9. Some lemmas on representations.

Let V be normal in G, $S = G/V$. Let N be an irreducible G-module (complex representation space), $M \subset N$ an irreducible V-module. Then $\sum_{g \in G} gM \subset N$ and is invariant under G, hence $\sum_{g \in G} gM = N$. Now each gM is an irreducible V-module. Hence we can find a subset $g_1, \ldots, g_m$ of elements of G such that

$$N = \sum_{i=1}^{m} g_i M \quad \text{as } \textit{direct sum}.$$

To see this consider a maximal subspace of N of the form $\sum_{i=1}^{m} g_i M$ (direct sum). If this is different from N, then some further gM exists which is not contained in $\sum_{i=1}^{m} g_i M$; since gM is irreducible $gM \cap \sum_{i=1}^{m} g_i M = 0$, and so $\sum_{i=1}^{m} g_i M$ is not maximal.

Let ρ, σ be respectively the isomorphism classes of N (as G-module) and M (as V-module). Then if $i^* : R(G) \to R(V)$ is the restriction homomorphism, we have

$$i^*(\rho) = \sum_{i=1}^{m} s_i(\sigma), \tag{1}$$

where $s_i = g_i^{-1}V$. We have just to recall that S operates on R(V), and this operation is such that $s(\sigma)$ is the class of gM if $s = g^{-1}V$. In detail, if $s = hV$ a representative V-module for $s(\sigma)$ is given by defining a new V-module structure on M as follows

$$v[x] = hvh^{-1}.x, \qquad x \in M.$$

Hence $x \to h^{-1}.x$ defines an isomorphism of this new V-module structure on M with the original V-module structure on $h^{-1}M$.

Now S operates trivially on R(G) and so $i^*(\rho)$ must be invariant under S. But

by (1) every irreducible component of $i^*(\rho)$ is of the form $s(\sigma)$ for some $s\in S$. Hence we must have $i^*(\rho)=n(\sum_j \sigma_j)$, where $\{\sigma_j\}$ is the complete set of (distinct) conjugates of σ (i.e. the "orbit" $S\sigma$). We state this as a lemma.

Lemma **(9.1)**. — *Let* V *be normal in* G, ρ *an irreducible representation of* G. *Then*

$$i^*(\rho)=n\Sigma\sigma_j,$$

where $\{\sigma_j\}$ *is a complete set of conjugate irreducible representations of* V.

Lemma **(9.2)**. — *Let* V *be normal in* G *with* $G/V=S$. *Let* σ *be an irreducible representation of* V, S_σ *the stabilizer of* σ *in* S. *Let*

$$i_*(\sigma)=\sum_k m_k\rho_k$$

be the decomposition of $i_*(\sigma)$ *(the induced representation) into irreducible representations* ρ_k *of* G. *Then*

$$\Sigma m_k^2=(S_\sigma:1).$$

Proof. — From $i_*(\sigma)=\Sigma m_k\rho_k$ we have

(2) $$(S:1).\dim\sigma=\Sigma m_k\dim\rho_k.$$

By Frobenius's theorem and (9.1) we have $i^*(\rho_k)=m_k(\Sigma\sigma_j)$, where $\{\sigma_j\}$ is the complete set of conjugates of σ. Hence

(3) $$\dim\rho_k=(S:S_\sigma).m_k.\dim\sigma.$$

From (2) and (3) we deduce

$$(S:1)=(S:S_\sigma)\,\Sigma m_k^2,$$

and so

$$(S_\sigma:1)=\Sigma m_k^2.$$

Lemma **(9.3)**. — *Let* V *be normal in* G *with* $G/V=S$. *Let* σ *be an irreducible representation of* V *with stabilizer* S_σ, *and let* $\{\sigma_j\}$ *be the complete set of conjugates of* σ. *Suppose that* $(S_\sigma:1)$ *is square-free. Then*

$$\Sigma\sigma_j\in i^*R(G).$$

Proof. — From (9.2) we have $\Sigma m_k^2=(S_\sigma:1)$. Since $(S_\sigma:1)$ is square-free this implies that the m_k have no common factor. Hence there exist integers a_k such that $\Sigma a_k m_k=1$. Hence

$$i^*(\Sigma a_k\rho_k)=(\Sigma a_k m_k)(\Sigma\sigma_j)=\Sigma\sigma_j.$$

Lemma **(9.4)**. — *Let* V *be normal in* G *with* $G/V=S$. *Suppose that* $(S:1)$ *is square-free. Then*

$$R(V)^S=i^*R(G),$$

where $R(V)^S$ *denotes the invariants of* S.

Proof. — We have already remarked that $i^*R(G)\subset R(V)^S$. Now a **Z**-basis for $R(V)^S$ is given by the sums of complete sets of conjugates $\Sigma\sigma_j$. But for any σ, since $S_\sigma\subset S$ and $(S:1)$ is square-free, it follows that $(S_\sigma:1)$ is square-free. Hence (9.4) follows at once from (9.3).

The special case of (9.4) which we shall need later is explicitly:

Proposition **(9.5)**. — *Let* V *be normal in* G *with* $G/V=Z_q$ *cyclic of prime order* q. *Then*

$$R(V)^{Z_q}=i^*R(G).$$

§ 10. Solvable groups.

In this section we shall prove (7.2) for solvable groups. The main step is the following:

Proposition (**10.1**). — *Let* V *be normal in* G *with* $G/V = Z_q$ *(q prime). Suppose* $\hat{\alpha}_V : \widehat{R(V)} \to \mathscr{K}^*(B_V)$ *is an isomorphism. Then* $\hat{\alpha}_G : \widehat{R(G)} \to \mathscr{K}^*(B_G)$ *is an isomorphism.*

Proof. — By (5.1) we have a strongly convergent spectral sequence:

$$H^*(Z_q, \mathscr{K}^*(B_V)) \Rightarrow \mathscr{K}^*(B_G)_{Z_q}.$$

By hypothesis $\mathscr{K}^*(B_V) \cong \widehat{R(V)}$, so that

$$E_2^p \cong H^p(Z_q, \widehat{R(V)}) \cong H^p(Z_q, R(V))\hat{} \quad \text{by } (3.17). \tag{1}$$

Here we must observe the following: $\widehat{R(V)}$ is the completion in the I(V)-adic topology. By (6.1) this is the same as the I(G)-adic topology, R(V) being viewed as R(G)-module. Since $i^*R(G) \subset R(V)^{Z_q}$ it follows that R(V) is an R(G)-Z_q-module as required for (3.17). Moreover $H^p(Z_q, R(V))\hat{}$ denotes the I(G)-adic completion.

Let $\xi_1, \ldots, \xi_n$ be the (classes of) irreducible representations of V. Suppose ξ_i for $1 \leqslant i \leqslant r$ are invariant under Z_q, and that the ξ_i for $i > r$ fall into sets of q conjugates (these are the only possibilities since q is prime). Then as a Z_q-module

$$R(V) = \mathbf{Z}\xi_1 \oplus \mathbf{Z}\xi_2 \oplus \ldots \oplus \mathbf{Z}\xi_r \oplus M, \tag{2}$$

where M is a free Z_q-module. Hence $H^{2k+1}(Z_q, R(V)) = 0$, and so $E_2^{2k+1} = 0$. Since $\widehat{R(V)} \cong \mathscr{K}^0(B_G)$ and $\mathscr{K}^1(B_G) = 0$ (by assumption) it follows from (5.4), and (1), that

$$G\mathscr{K}^*(B_G)_{Z_q} \cong H^*(Z_q, R(V))\hat{}. \tag{3}$$

To prove $\hat{\alpha}_G$ an isomorphism it is only necessary, by (7.9), to show that $\alpha(R(G))$ is dense in $\mathscr{K}^*(B_G)$. Since $\alpha(R(G)) \subset \hat{\alpha}\widehat{R(G)}$, and since the Z_q-topology of $\mathscr{K}^*(B_G)$ is finer than its G-topology it will be sufficient to prove that $\hat{\alpha}(\widehat{R(G)})$ is dense in $\mathscr{K}^*(B_G)$ *for the* Z_q*-topology.* This means we have to prove, for each p, that

$$G^p\widehat{R(G)} \to G^p\mathscr{K}^*(B_G)_{Z_q} \tag{4}$$

is an epimorphism, where we give $\widehat{R(G)}$ the induced filtration: $\widehat{R_p(G)} = \hat{\alpha}_G^{-1}\mathscr{K}_p^*(B_G)_{Z_q}$. For $p = 0$ we have to show (using 3)) that

$$\widehat{R(G)} \to (R(V)^{Z_q})\hat{} \to 0 \tag{5}$$

is exact. But this follows from the fact that

$$R(G) \to R(V)^{Z_q} \to 0 \tag{6}$$

is exact (9.5), and that I(G)-adic completion is an exact functor (3.16). For $p = 2k+1$ it is trivial. Suppose therefore $p = 2k$, $k > 0$. To prove (4) in this case it will be sufficient, using (5), to prove that

$$\lambda : G^{2k}\widehat{R(Z_q)} \otimes_{\mathbf{Z}} (R(V)^{Z_q})\hat{} \to G^{2k}\mathscr{K}^*(B_G)_{Z_q}$$

is an epimorphism, where λ is defined using the module multiplication of (2.7). But we have a commutative diagram:

$$\begin{array}{ccc} H^{2k}(Z_q, \mathbf{Z}) \otimes_{\mathbf{Z}} R(V)^{Z_q} & \xrightarrow{\mu} & H^{2k}(Z_q, R(V)) \\ \downarrow & & \downarrow \tau \\ H^{2k}(Z_q, \mathbf{Z}) \otimes_{\mathbf{Z}} (R(V)^{Z_q})^{\wedge} & \xrightarrow{\lambda} & H^{2k}(Z_q, R(V))^{\wedge}, \end{array}$$

where we have substituted for $G^{2k}\widehat{R(Z_q)}$ and $G^{2k}\mathscr{K}^*(B_G)$ by (8.1) and (3), and where λ, μ are now the obvious maps (as follows from (2.7)). From (2) we see that μ is an epimorphism, and the finiteness of $H^{2k}(Z_q, R(V))$ implies, by (3.3), that τ is an epimorphism. Hence λ is an epimorphism as required. This completes the proof.

Proposition **(10.2)**. — *Let* G *be a solvable group. Then* $\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$ *is an isomorphism.*

Proof. — A solvable group has, by definition, a composition series

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_{n-1} \supset G_n = 1.$$

With G_{i-1}/G_i cyclic of prime order. The length n depends only on G. We prove (10.2) by induction on n. For $n=1$, G is cyclic and so the result follows from (8.1). Suppose it is true for groups of length $n-1$. If G is of length n, then G_1 has length $n-1$. Hence $\hat{\alpha}_{G_1}$ is an isomorphism and $G/G_1 = Z_q$ (q prime). Hence by (10.1) $\hat{\alpha}_G$ is an isomorphism.

§ 11. The "completion" of Brauer's theorem.

We recall that an *elementary group* is a product of a p-group and a cyclic group. In particular an elementary group is solvable.

Let G be a finite group, $\{H_\lambda\}_{\lambda \in \Lambda}$ the family of all elementary subgroups of G. Let Q denote the group of inner automorphisms of G. Then Q operates on Λ, $\sigma(\lambda) \in \Lambda$ being defined for $\sigma \in Q$, $\lambda \in \Lambda$ by $H_{\sigma(\lambda)} = \sigma H_\lambda$. For any (ordered) triple λ, μ, $\nu \in \Lambda$ we define a homomorphism

$$\Theta_{\lambda,\mu,\nu} : R(H_\lambda) \to R(H_\mu \cap H_\nu)$$

as follows:

$$\Theta_{\lambda,\mu,\nu} = \begin{cases} \text{restriction if } \lambda = \mu,\ \lambda \neq \nu \\ -\text{restriction if } \lambda = \nu,\ \lambda \neq \mu \\ 0 \text{ otherwise.} \end{cases}$$

For any triple λ, μ, σ with λ, $\mu \in \Lambda$ and $\sigma \in Q$ we define a homomorphism

$$\Phi_{\lambda,\mu,\sigma} : R(H_\lambda) \to R(H_{\sigma(\mu)})$$

as follows
$$\Phi_{\lambda,\mu,\sigma} = \begin{cases} 1 & \text{if } \lambda=\sigma(\mu),\ \lambda \neq \mu \\ -\sigma^* & \text{if } \lambda \neq \sigma(\mu),\ \lambda=\mu \\ 1-\sigma^* & \text{if } \lambda=\mu=\sigma(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

The set of all $\Theta_{\lambda,\mu,\nu}$ and $\Phi_{\lambda,\mu,\sigma}$ defines a homomorphism

$$\Psi : \sum_{\lambda\in\Lambda} R(H_\lambda) \to \sum_{\mu,\nu\in\Lambda} R(H_\mu \cap H_\nu) \oplus \sum_{\substack{\mu\in\Lambda\\ \sigma\in Q}} R(H_{\sigma(\mu)}).$$

We shall now examine the kernel of Ψ. For any pair $\mu, \nu\in\Lambda$ and $\rho=\sum_\lambda \rho_\lambda \in \sum_\lambda R(H_\lambda)$, the component $\Psi(\rho)_{\mu,\nu}$ is by definition

$$\Psi(\rho)_{\mu,\nu} = \sum_\lambda \Theta_{\lambda,\mu,\nu}(\rho_\lambda),$$
$$= r_{\mu,\nu}(\rho_\mu) - r_{\nu,\mu}(\rho_\nu),$$

where $r_{\mu,\nu}$ is the restriction $R(H_\mu) \to R(H_\mu \cap H_\nu)$.

Next, for any pair $\pi\in\Lambda$, $\sigma\in Q$ the component $\Psi(\rho)_{\pi,\sigma}$ is by definition

$$\Psi(\rho)_{\pi,\sigma} = \sum_\lambda \Phi_{\lambda,\pi,\sigma}(\rho_\lambda),$$
$$= \rho_{\sigma(\pi)} - \sigma^*(\rho_\pi).$$

Hence $\rho\in\operatorname{Ker}\Psi$ if and only if

(i) $r_{\mu,\nu}(\rho_\mu) = r_{\nu,\mu}(\rho_\nu)$ for all $\mu, \nu\in\Lambda$,

(ii) $\rho_{\sigma(\pi)} = \sigma^*(\rho_\pi)$ for all $\pi\in\Lambda$, $\sigma\in Q$.

Consider now the character $\chi_\lambda = \chi(\rho_\lambda)$. It is a function on H_λ. (i) and (ii) are equivalent to:

(i′) $\chi_\mu = \chi_\nu$ in $H_\mu \cap H_\nu$, for all $\mu, \nu\in\Lambda$,

(ii′) $\chi_{\sigma(\pi)} = \sigma^*(\chi_\pi)$ for all $\pi\in\Lambda$, $\sigma\in Q$.

From (i′) the set of χ_λ defines a single-valued function (with values in $\mathbf{C}$) on $\bigcup_\lambda H_\lambda$. Since the family of elementary groups includes all cyclic groups it follows that $\bigcup_\lambda H_\lambda = G$. Thus we have a function χ on G. Then (ii′) asserts that $\sigma^*(\chi)=\chi$ for all $\sigma\in Q$. Hence $\rho=\Sigma\rho_\lambda$ belongs to the kernel of Ψ if and only if $\chi_\lambda=\chi(\rho_\lambda)$ is, for all λ, the restriction to H_λ of a class function χ on G. But the theorem of Brauer [5, Theorem B] asserts that such a class function χ is necessarily a character of G. Thus we may reformulate Brauer's theorem as follows:

Lemma (**11.1**). — *We have an exact sequence*

$$0 \to R(G) \xrightarrow{r} \sum_{\lambda\in\Lambda} R(H_\lambda) \xrightarrow{\Psi} \sum_{\mu,\nu\in\Lambda} R(H_\mu \cap H_\nu) \oplus \sum_{\substack{\pi\in\Lambda\\ \sigma\in Q}} R(H_{\sigma(\pi)})$$

where r is the restriction, $\{H_\lambda\}_{\lambda\in\Lambda}$ is the family of all elementary subgroups of G *and* Ψ *is the homomorphism defined above.*

Lemma **(11.2).** — *In the notation of* (11.1) *we have an exact sequence.*

$$0 \to \widehat{R(G)} \xrightarrow{\hat{r}} \sum_{\lambda} \widehat{R(H_\lambda)} \xrightarrow{\hat{\Psi}} \sum_{\mu,\nu} R(H_\mu \cap H_\nu)^\wedge \oplus \sum_{\pi,\sigma} R(H_{\sigma(\pi)})^\wedge$$

where each completion is with respect to the augmentation topology.

Proof. — This follows at once from (11.1), (6.1) and (3.16).

Proposition **(11.3).** — *For any finite group*

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$$

is an epimorphism.

Proof. — With the notation introduced above we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{R(G)} \arrow[r] \arrow[d, "\hat{\alpha}_G"] & \Sigma\widehat{R(H_\lambda)} \arrow[r, "\Psi"] \arrow[d, "\beta"] & \Sigma R(H_\mu \cap H_\nu)^\wedge \oplus \Sigma R(H_{\sigma(\pi)})^\wedge \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathscr{K}^*(B_G) \arrow[r, "\rho"] & \Sigma\mathscr{K}^*(B_{H_\lambda}) \arrow[r, "\delta"] & \Sigma\mathscr{K}^*(B_{H_\mu \cap H_\nu}) \oplus \Sigma\mathscr{K}^*(B_{H_{\sigma(\pi)}})
\end{tikzcd}$$

The top line is exact by (11.2). In the bottom line we have exactness at $\mathscr{K}^*(B_G)$, i.e. ρ a monomorphism, by (4.10) and the fact that the H_λ include all Sylow subgroups of G. δ is defined in a precisely analogous way to Ψ, and since Q (the group of inner automorphisms of G) operates trivially on $\mathscr{K}^*(B_G)$ (cf. § 4) it follows that $\delta\rho = 0$. Now β is an isomorphism, since the H_λ are solvable (10.2). Also γ is a monomorphism (8.4). Hence $\rho\hat{\alpha}_G : \widehat{R(G)} \to \operatorname{Ker}\delta$ is an epimorphism. Since $\rho\mathscr{K}^*(B_G) \subset \operatorname{Ker}\delta$, it follows that

$$\rho\mathscr{K}^*(B_G) = \operatorname{Ker}\delta = \rho\hat{\alpha}_G(\widehat{R(G)}).$$

Since ρ is a monomorphism this implies $\mathscr{K}^*(B_G) = \hat{\alpha}_G(\widehat{R(G)})$, i.e. $\hat{\alpha}_G$ is an epimorphism.

(11.3) and (8.4) together complete the proof of the main theorem (7.2) (in view of (7.9)).

§ 12. The filtration of R(G).

As remarked in the introduction the filtration on R(G) has been defined topologically, via B_G, and the problem of giving an algebraic definition of the filtration is left unsolved. There is however a good candidate for such an algebraic definition due to Grothendieck, which we shall proceed to describe.

We recall first the notion of a λ-*ring*, introduced by Grothendieck. A λ-ring is a commutative ring R (with identity) with operators

$$\lambda^i : R \to R$$

(i an integer $\geqslant 0$), satisfying the following conditions:

(1) $$\lambda^0 x = 1,\ \lambda^1 x = x,\ \lambda^n(x+y) = \sum_{i=0}^{n} \lambda^i(x).\lambda^{n-i}(y).$$

If we put, for any $x \in \mathrm{R}$,

(2) $$\lambda_t(x) = \sum_{n\geqslant 0} \lambda^n(x)t^n \in \mathrm{R}[[t]],$$

the relations (1) express the fact that $x \rightarrow \lambda_t(x)$ is a homomorphism of the additive group R into the multiplicative group $1+\mathrm{R}[[t]]^+$ (formal power series with constant term 1) which is a right inverse of the homomorphism $(1+\sum_{i\geqslant 1} x_i t^i) \rightarrow x_1$.

The ring **Z** of integers has a unique λ-ring structure such that

(3) $$\lambda_t(1) = 1+t.$$

Then we have

(4) $$\lambda_t(n) = (1+t)^n,\ \lambda^i(n) = \binom{n}{i}.$$

An *augmented λ-ring* will then mean a λ-ring R together with a homomorphism $\varepsilon : \mathrm{R} \rightarrow \mathbf{Z}$ of λ-rings, **Z** having the λ-structure just described.

If R is any λ-ring, Grothendieck defines operators γ^n by the formula:

(5) $$\gamma^n(x) = \lambda^n(x+n-1),$$

and $\gamma_t(x)$ by

(6) $$\gamma_t(x) = \sum_{n\geqslant 0} \gamma^n(x)t^n.$$

Then γ_t and λ_t are related by the formula

(7) $$\gamma_t(x) = \lambda_{t/1-t}(x),$$

or equivalently

(8) $$\lambda_s(x) = \gamma_{s/1+s}(x).$$

These show, in particular, that the γ^n also satisfy the identities (1).

Now let R be an augmented λ-ring and let $\mathrm{I} = \mathrm{Ker}\,\varepsilon$, where $\varepsilon : \mathrm{R} \rightarrow \mathbf{Z}$ is the augmentation. Then the filtration on R defined by Grothendieck is as follows [1]: R_{2n} *is the subgroup generated by the monomials*

$$\gamma^{n_1}(x_1).\gamma^{n_2}(x_2), \ldots, \gamma^{n_k}(x_k)$$

with $x_i \in \mathrm{I}$ *and* $\sum_1^k n_i \geqslant n$. We shall refer to this as the γ-*filtration* of the augmented λ-ring. Since ε commutes with λ_t it also, by (7), commutes with γ_t, and hence if $x \in \mathrm{I}$

$$\varepsilon\gamma^n(x) = \gamma^n\varepsilon(x) = \gamma^n(0) = 0 \qquad \text{for } n \geqslant 1.$$

This shows that, in the γ-filtration, we have

(9) $$\mathrm{R}_2 = \mathrm{I}, \qquad \mathrm{R}_0 = \mathrm{R}.$$

From the definition it is clear that the γ-filtration makes R a *filtered ring*, i.e.

$$\mathrm{R}_{2n}.\mathrm{R}_{2m} \subset \mathrm{R}_{2n+2m}.$$

---

[1] We adopt an "even" notation for the filtration in order to conform with the topological aspect.

As pointed out by Grothendieck the rings $K^0(X)$ (for a connected finite CW-complex X) and $R(G)$ are augmented λ-rings, the λ^i being the exterior powers. Thus if $x \in K^0(X)$ is represented by the formal sum

$$x = \Sigma n_i \xi_i,$$

where the ξ_i are vector bundles on X, $\lambda_t(x)$ is defined by

$$\lambda_t(x) = \Pi \lambda_t(\xi_i)^{n_i},$$

where $\lambda^k(\xi_i)$ is the k-th exterior power of the vector bundle ξ_i. It is not difficult to show that this definition of $\lambda_t(x)$ is unambiguous, and makes $K^0(X)$ an augmented λ-ring. In a similar way if $\rho \in R(G)$ is given by

$$\rho = \Sigma n_i \rho_i,$$

where the ρ_i are the irreducible representations of G, then $\lambda_t(\rho)$ is defined by

$$\lambda_t(\rho) = \Pi \lambda_t(\rho_i)^{n_i}$$

where $\lambda^k(\rho_i)$ is the k-th exterior power of the representation ρ_i. Since $R(G)$ is a free abelian group with the ρ_i as basis, there is in this case nothing to prove. It is perhaps worth remarking that if we identify $R(G)$ with the character ring of G, so that $R(G)[[t]]$ becomes a subring of the ring of all functions $G \to \mathbf{C}[[t]]$, then for any representation ρ of G, $\lambda_t(\rho)$ is the function given by

$$g \to \det(1 + t\rho(g)).$$

We shall now consider the γ-filtration of $R(G)$, and to distinguish it from the topological filtration we shall denote the subgroups of the γ-filtration by $R'_{2n}(G)$.

Proposition (**12.1**). — *Let* G *be a finite group,* $\rho_1, \ldots, \rho_k$ *its irreducible representations. Put* $\sigma_{ij} = \lambda^i(\rho_j - \varepsilon(\rho_j) + i - 1)$ *and define the weight of* σ_{ij} *to be* i. *Then* $R'_{2n}(G)$, *the n-th subgroup of the* γ*-filtration of* $R(G)$, *is the subgroup generated by the monomials of weight* $\geqslant n$ *in the elements* $\sigma_{ij}(i = 1, 2, \ldots, \varepsilon(\rho_j); j = 1, \ldots, k)$.

Proof. — The elements $\rho_j - \varepsilon(\rho_j)$ form an additive base for $I(G)$. Now by (7) it follows that $\gamma^i(nx + my)$ is expressible as a polynomial of weight i in the $\gamma^k(x), \gamma^k(y)$ (where weight $\gamma^k = k$) for all integers m, n. Hence, from the definition of $R'_{2n}(G)$, we see that it is generated additively by the monomials of weight $\geqslant n$ in the σ_{ij}. However for $i > \varepsilon(\rho_j)$ we have

$$\sigma_{ij} = \lambda^i(\rho_j + k)$$

where $k \geqslant 0$ and $i > \varepsilon(\rho_j) + k$, so that $\sigma_{ij} = 0$. This completes the proof.

Corollary (**12.2**). — *The graded ring associated to the* γ*-filtration of* $R(G)$ *is finitely-generated. The number of generators can be chosen equal to the sum of the dimensions of the irreducible representations of* G.

The γ-filtration of $R(G)$ defines a topology which we shall call the γ-*topology*.

Corollary (**12.3**). — *The* γ*-topology of* $R(G)$ *coincides with the* $I(G)$*-adic topology.*

Proof. — From the definition we have

$$I(G)^n \subset R'_{2n}(G).$$

Conversely, let n be given, and put $m = sn$ where $s = \max \varepsilon(\rho_j)$. Consider any monomial in the σ_{ij} of weight $\geqslant m$. Since $i \leqslant s$, the degree of the monomial must be $\geqslant \frac{m}{s} = n$, where we define the degree of each σ_{ij} to be 1. But by 9) $\sigma_{ij} \in I(G)$. Hence

$$R'_{2m}(G) \subset I(G)^n,$$

completing the proof.

Corollary **(12.4)**. — *For an abelian group* G, *we have*

$$R'_{2n}(G) = I(G)^n$$

Proof. — If G is abelian all the irreducible representations ρ_j have dimension 1. Hence, by (12.1), $R'_{2n}(G)$ is generated by the monomials in the elements $\sigma_{1j} = (\rho_j - 1)$ of weight $\geqslant n$. Since the σ_{1j} form an additive basis for $I(G)$ we have

$$R'_{2n}(G) = I(G)^n$$

as required.

We turn next to consider the γ-filtration of the augmented ring $K^0(X)$. Again to distinguish this from the topological filtration we shall denote the subgroups by $K^{0\prime}_{2n}(X)$.

Proposition **(12.5)**. — *For all n we have*

$$K^{0\prime}_{2n}(X) \subset K^0_{2n}(X).$$

Proof. — Since $K^0(X)$, with the topological filtration, is a filtered ring it will be sufficient to show that if $x \in K^0(X)$ with $\varepsilon(x) = 0$ then

$$\lambda^n(x + n - 1) \in K^0_{2n}(X).$$

In view of (2.2) it will be sufficient to show that, if $\dim X \leqslant 2(n-1)$, then

$$\lambda^n(x + n - 1) = 0.$$

Now since $\varepsilon(x+n-1) = n-1 \geqslant \frac{1}{2} \dim X$ it follows that $x+n-1$ is in the "stable range" and so (it is easy to show) can be represented by a vector bundle ξ of dimension $n-1$. Then $\lambda^n(x+n-1)$ is represented by $\lambda^n(\xi)$ and this is zero since $n > \dim \xi$.

Since $K^0(X)$ is an augmented λ-ring, for all finite connected CW-complexes X, it follows that the inverse limit group $\mathscr{K}^0(B_G)$ is also an augmented λ-ring. Moreover from the definitions it is immediate (cf. § 1) that

$$\alpha : R(G) \to \mathscr{K}^0(B_G)$$

is a homomorphism of augmented λ-rings, and hence $\alpha(R'_{2n}(G)) \subset \mathscr{K}^{0\prime}_{2n}(B_G)$. From (12.5) therefore we deduce

Proposition **(12.6)**. — *Let* $\{R'_{2n}(G)\}$ *be the γ-filtration of the augmented λ-ring* R(G), *and let* $\{R_{2n}(G)\}$ *be the topological filtration. Then, for all n, we have*

$$R'_{2n}(G) \subset R_{2n}(G).$$

Next we need an elementary lemma.

Lemma **(12.7)**. — *In any λ-ring, with γ being defined by* (5), *we have the identity*

$$\lambda^n(x) = \sum_{i=0}^{n} \gamma^i(x-n).$$

Proof. — By (1) and (7)

$$\begin{aligned}\lambda_t(x) &= \lambda_t(x-n)\,.\,\lambda_t(n) = \gamma_{t/1+t}(x-n)\,.\,(1+t)^n \\ &= \sum_{i\geq 0} \gamma^i(x-n)t^i(1+t)^{n-i}.\end{aligned}$$

Equating coefficients of t^n, the lemma follows.

Proposition **(12.8)**. — *For* $n=0, 1, 2$ *we have*

$$R'_{2n}(G) = R_{2n}(G).$$

Proof. — In view of (9) we need only consider the case $n=2$. Let $y \in R_4(G)$, then we can write $y=\rho-\tau$, where ρ, τ are representations of G of dimension n and by (7.7), $\lambda^n(\rho)=\lambda^n(\tau)$. Thus

$$y = \{\rho-\lambda^n(\rho)-n+1\}-\{\tau-\lambda^n(\tau)-n+1\}$$

Applying (12.7) with ρ, τ instead of x we see that

$$\rho-\lambda^n(\rho)-n+1 = -\sum_{i=2}^{n} \gamma^i(\rho-n)$$

$$\tau-\lambda^n(\tau)-n+1 = -\sum_{i=2}^{n} \gamma^i(\tau-n).$$

This shows that $y \in R'_4(G)$ which, in view of (12.6), completes the proof.

The preceding results make it not unreasonable to conjecture that, for all n and G, we have $R'_{2n}(G)=R_{2n}(G)$. We shall in fact verify this conjecture in the next section for a few explicit groups.

In connection with this conjecture, (12.2) should be compared with a recent result of L. Evens, to the effect that $H^*(G, \mathbf{Z})$ is finitely-generated [1].

§ 13. Some examples.

In this section we shall compute a few illustrative examples of the spectral sequence $H^*(G, \mathbf{Z}) \Rightarrow \widehat{R(G)}$.

The symmetric group S_3.

The character table of S_3 is

(Conjugacy classes)

(Irreducible representations)	1^3	2 1	3
1	1	1	1
x	1	-1	1
y	2	0	-1

[1] (Added in proof) It can in fact be proved that $G\mathscr{K}^*(B_G)$ is finitely generated.

Now by (4.9) the filtration on $R = R(S_3)$ is determined by its p-Sylow subgroups ($p=2, 3$). Since these are cyclic (8.1) enables us to determine the filtration completely. Putting $\alpha = 1 - x$, $\beta = 2 - y$ we find

$$R_2 = \{\alpha, \beta\}, \qquad R_4 = \{2\alpha, \alpha + \beta\}$$
$$R_{4k+4} = (\alpha+\beta) R_{4k} \quad (k \geqslant 1),$$

where for R_2, R_4 we have written the generators.

For the cohomology of S_3 it is well-known that we have

$$H^{4k+2}(S_3, \mathbf{Z}) = Z_2, \qquad H^{2q+1}(S_3, Z) = 0$$
$$H^{4k+4}(S_3, \mathbf{Z}) = Z_6,$$

and that the generator of $H^4(S_3, \mathbf{Z})$ gives the periodicity (by cup-products).

Since there are only even dimensions the spectral sequence collapses and we have (5.3) a ring isomorphism $H^*(S_3, \mathbf{Z}) \cong GR(S_3)$. This checks with the above formulae, α mod R_4 giving the generator of $H^2(S_3, \mathbf{Z})$ and $(\alpha+\beta)$ mod R_6 giving the generator of $H^4(S_3, \mathbf{Z})$.

Since $\lambda^2 y = x$, the elements σ_{ij} of (12.1) which generate the γ-filtration of $R(G)$ are

$$-\alpha, -\beta \in R'_2(G), -\alpha + \beta \in R'_4(G).$$

Since $\alpha^2 = 2\alpha$, $\beta^2 = 3\beta - \alpha$, $\alpha\beta = 2\alpha$ it follows that $R'_{2n}(G) = R_{2n}(G)$ for all n, in accordance with the conjecture of § 12.

The Quaternion group.

G is now the group whose elements are ± 1, $\pm i$, $\pm j$, $\pm k$ under quaternion multiplication.

The character table of G is

(Conjugacy classes)

(Irreducible representations)		1	-1	$\pm i$	$\pm j$	$\pm k$
	1	1	1	1	1	1
	x_i	1	1	1	-1	-1
	x_j	1	1	-1	1	-1
	x_k	1	1	-1	-1	1
	y	2	-2	0	0	0

The cohomology of G is [7, p. 254]

$$H^{4k+2}(G, \mathbf{Z}) = Z_2 \oplus Z_2,$$
$$H^{4k+4}(G, \mathbf{Z}) = Z_8,$$
$$H^{2q+1}(G, \mathbf{Z}) = 0,$$

and the generator of $H^4(G, \mathbf{Z})$ gives the periodicity.

Thus again the spectral sequence is trivial and so we have a ring isomorphism $H^*(G, \mathbf{Z}) \cong GR(G)$. However, unlike the case of S_3, we have no guaranteed method of determining the filtration on $R(G)$. In fact we *can* determine the filtration as follows.

First put $\alpha = 1 - x_i$, $\beta = 1 - x_i$, $\gamma = 3 - x_i - x_j - x_k$, $\delta = 2 - y$. Then $R_2 = \{\alpha, \beta, \gamma, \delta\}$ and the products are given by:

$$\begin{aligned}
&\alpha^2 = 2\alpha, \quad \alpha\beta = 2\alpha + 2\beta + \gamma, \quad \alpha\gamma = 4\alpha, \quad \alpha\delta = 2\alpha,\\
&\beta^2 = 2\beta \qquad\qquad\qquad\qquad \beta\gamma = 4\beta, \quad \beta\delta = 2\beta,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \gamma^2 = 4\gamma, \quad \gamma\delta = 2\gamma,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \delta^2 = 2\delta - \gamma.
\end{aligned}$$

Now to determine R_4 it is sufficient by (7.7) to consider determinants (or the first Chern class). It is easy to see that $c_1(\alpha) = a$, $c_1(\beta) = b$ are generators of $H^2(G, \mathbf{Z})$, and that $c_1(\gamma) = 0$. Since there is an automorphism of G permuting x_i, x_j, x_k cyclically, it follows by symmetry that $\det y = 1$, i.e. $c_1(\delta) = 0$. Thus $R_4 = \{2\alpha, 2\beta, \gamma, \delta\}$. From the product formulae we find

$$R_2R_4 = \delta R_2 = \{2\alpha, 2\beta, 2\gamma, 4\delta - \gamma\}.$$

This is of index 8 in R_4 and so must be R_6. Moreover δ mod R_6 gives a generator d of $H^4(G, \mathbf{Z})$. The fact that d gives the periodicity of $H^*(G, \mathbf{Z})$ then shows that the filtration of $R(G)$ is given by

$$R_{4k+2} = \delta^k R_2, \; R_{4k+4} = \delta^k R_4.$$

Since $\lambda^2 y = 1$, the elements σ_{ij} of (12.1) are

$$-\alpha, -\beta, \alpha + \beta - \gamma \in R_2'(G), \; -\delta \in R_4'(G).$$

Since $x_i x_j x_k = 1$ we deduce

$$(1-\alpha)(1-\beta)(1+\alpha+\beta-\gamma) = 1$$

and hence $\gamma \in I(G)^2 \subset R_4'(G)$. Also $\alpha^2 = 2\alpha$, $\beta^2 = 2\beta$, so that we have $R_{2n}'(G) = R_{2n}(G)$ for all n, in accordance with our conjecture.

We can now use the product formulae in $R(G)$ to compute cup-products. We get

$$a^2 = b^2 = 0, \qquad ab = 4d.$$

Remark. — Whenever, as in this example, the odd cohomology groups vanish *and* the filtration on $R(G)$ is known the cup-products in $H^*(G, \mathbf{Z})$ can be read off from the character table of G.

A product of cyclic groups of order 2.

Let $G = Z_2 \times Z_2 \times \ldots \times Z_2$ (n factors). Then $H^*(G, \mathbf{Z})$ has non-zero odd-dimensional groups, so that the spectral sequence does not collapse. Now the first operator d_3 of the spectral sequence is the Steenrod operation Sq^3 (2.4), *d*). A direct

calculation ([1]) shows that $E_4 = H(H^*(G, \mathbf{Z}), d_3)$ is generated multiplicatively over Z_2 by elements x_i (of dimension 2) $i = 1, \ldots, n$ with relations

$$x_i^2 x_j = x_i x_j^2 \ (i, j = 1, \ldots, n). \tag{1}$$

Thus E_4 has only even-dimensional terms and so (cf. (5.3)) $E_4 \cong E_\infty \cong GR(G)$.

On the other hand $R(G)$ is generated over $\mathbf{Z}$ by elements ρ_i $(i = 1, \ldots, n)$ with $\rho_i^2 = 1$. Putting $\alpha_i = 1 - \rho_i$, we get the relations $\alpha_i^2 = 2\alpha_i$. These imply the equations

$$\alpha_i^2 \alpha_j = 2\alpha_i \alpha_j = \alpha_i \alpha_j^2 \tag{2}$$

α_i mod R_4 gives an element of E_4 which is easily seen to be x_i. The relations (2) then check with the relations (1). Moreover we see that $R_{2n}(G) = I(G)^n$ which, in view of (12.4), agrees with our conjecture.

Remark. — The calculation for a product of cyclic groups Z_p (p a prime $\neq 2$) is quite similar. One has to use the operator d_{2p-1} of the spectral sequence.

A direct description of the filtration on $R(G)$, for example a proof of the conjecture of § 12, would lead to lower bounds for the cohomology groups of G. In the absence of such a description we can only give a weak qualitative result in this direction.

Theorem **(13.1)**. — *Let* G *be a finite group containing more than one element. Then there exist arbitrarily large integers n so that* $H^n(G, \mathbf{Z}) \neq 0$.

Proof. — The hypothesis on G and (6.10) imply that

$$\mathrm{Im}\{I(G) \to \widehat{I(G)}\} = I(G) / \bigcap_{n=1}^{\infty} I(G)^n$$

is a free abelian group of rank > 0. Now if $H^n(G, \mathbf{Z}) = 0$ for all sufficiently large n then by (7.6) $\widehat{I(G)}$ would be finite. This gives a contradiction, and so the theorem is proved.

APPENDIX

Chern Classes.

If ξ is an n-dimensional complex vector bundle over a CW-complex X, then ξ has Chern classes $c_i(\xi) \in H^{2i}(X, \mathbf{Z})$. For the definition and properties of these classes we refer to [9, § 4] or [2, § 9]. Taking $X = B_G$, the classifying space of a finite group G, we deduce

(1) *To each complex representation* ρ *of* G *there are associated Chern classes* $c_i(\rho) \in H^{2i}(G, \mathbf{Z})$, $c_0(\rho) = 1$ *and* $c_i(\rho) = 0$ *for* $i > \dim \rho$.

The Chern classes $c_i(\rho)$ are thus defined topologically. It would be highly desirable to have a direct algebraic definition of them, but like the corresponding problem for the spectral sequence $H^*(G, \mathbf{Z}) \Rightarrow \widehat{R(G)}$ this is still unsolved.

We proceed now to give the formal properties of Chern classes.

(2) *If* $f : G' \to G$ *is a homomorphism and* ρ *is a representation of* G, *then*

$$c_i(f^*\rho) = f^* c_i(\rho).$$

([1]) I am indebted to C.T.C. Wall for this calculation.

(3) *For* 1*-dimensional representations*

$$c_1 : \mathrm{Hom}(G, U(1)) \to H^2(G, \mathbf{Z})$$

is an isomorphism.

This is in fact the isomorphism (1.3).

(4) *If* ρ^* *is the dual (or contragredient) representation of* ρ, *then*

$$c_i(\rho^*) = (-1)^i c_i(\rho).$$

It is convenient to consider the sum of all the Chern classes

$$c(\rho) = \sum_i c_i(\rho) \in H^*(G, \mathbf{Z}).$$

(5) $c(\rho \oplus \sigma) = c(\rho).c(\sigma)$.

Suppose now that $\mu : U(n) \to U(m)$ is a representation of the unitary group $U(n)$. Then μ has associated with it m integral linear forms $w_1, \ldots, w_m$ (the weights) in variables $x_1, \ldots, x_n$ (cf. [2, § 10]). Moreover the elementary symmetric functions $\sigma_i(w_1, \ldots, w_m)$ are symmetric in $x_1, \ldots, x_n$ and hence expressible as polynomials in the elementary symmetric function $\sigma_j(x_1, \ldots, x_n)$. Let $P_{\mu, i}$ be this polynomial, thus

$$\sigma_i(w_1, \ldots, w_m) = P_{\mu, i}(\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_n(x_1, \ldots, x_n)).$$

Now if $\rho : G \to U(n)$ is a representation of G then $\mu\rho : G \to U(m)$ is another representation. The relation between their Chern classes is given by [2, § 10].

(6) $c_i(\mu\rho) = P_{\mu, i}(c_1(\rho), \ldots, c_n(\rho))$.

In particular, taking μ to be the n-th exterior power representation $\lambda^n : U(n) \to U(1)$ we deduce

(7) *If* $\dim \rho = n$, *then*

$$c_1(\rho) = c_1(\lambda^n \rho).$$

In view of (3) this means that $c_1(\rho)$ is effectively known for any ρ.

If $x_1, \ldots, x_n, y_1, \ldots, y_m$ are two sets of indeterminates with elementary symmetric functions a_i, b_i respectively, we can define polynomials Q_k by the formula

$$\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (1 + t(x_i + y_j)) = \sum_k Q_k(a_1, \ldots, a_n, b_1, \ldots, b_m) t^k$$

where t is an indeterminate. The Chern classes of a tensor product are then given by

(8) $c_k(\rho \otimes \sigma) = Q_k(c_1(\rho), \ldots, c_n(\rho), c_1(\sigma), \ldots, c_m(\sigma))$.

Note that if $\dim \rho = \dim \sigma = 1$, (8) gives $c_1(\rho \otimes \sigma) = c_1(\rho) + c_1(\sigma)$ which is part of the assertion of (3).

In view of (5) the "total" Chern class c may be extended to give a homomorphism

$$c : R(G) \to A(G)$$

of the additive group R(G) into the multiplicative group A(G) consisting of elements of $\prod_{k=0}^{\infty} H^{2k}(G, \mathbf{Z})$ with constant term 1. For example suppose G is a cyclic group of order n, and let ρ be the basic 1-dimensional representation and x the corresponding generator of $H^2(G, \mathbf{Z})$ (cf. § 8). Then the elements of A(G) are formal power series

$$1 + \sum_{k=1}^{\infty} a_k x^k, \qquad a_k \in Z_n.$$

R(G) is a free abelian group generated by $1, \rho, \rho^2, \ldots, \rho^{n-1}$ and c is given by

$$c\left(\sum_{k=0}^{n-1} m_k \rho^k\right) = \prod_{k=0}^{n-1} (1 + kx)^{m_k},$$

where on the right k is regarded as an element of Z_n and if $m_k < 0$ we expand $(1 + kx)^{m_k}$ as a formal power series. If $n = p$ is a prime then Z_p is a field and so

$$\prod_{k=0}^{p-1} (1 + kx)^{m_k} = 1 \Leftrightarrow m_k = 0 \quad \text{for} \quad k \neq 0.$$

Thus, for G cyclic of *prime order*, $\qquad c : I(G) \to A(G)$

is a monomorphism. This certainly cannot hold for general cyclic groups, since $\theta : R(G) \to \widehat{R(G)}$ is not in general a monomorphism (6.10) and $\theta(\sigma) = 0 \Rightarrow c(\sigma) = 0$ (see below). However even in the case of a p-group, when θ is a monomorphism (6.11), it is still possible for $c : I(G) \to A(G)$ to have a non-zero kernel. As an example let G be cyclic of order p^2, then $c(p\rho^p - p) = (1+px)^p = 1.$

It appears unlikely that the formal properties of Chern classes listed above are sufficient to prove their uniqueness (working only within the category of finite groups). It is probable that one would have to add a formula for $c_i(i_*(\rho))$ where $i_*(\rho)$ is the "induced representation". However it is easy to see that there can be no simple formula involving only $c_i(\rho)$. For example let $i : G' \to G$ be the inclusion with $G = Z_p$ and G' the identity, and take $\rho = 1$. Then $i_*(\rho)$ is the regular representation $\sum_{k=0}^{p-1} \rho^k$ of G and

$$c\left(\sum_{k=0}^{p-1} \rho^k\right) = \prod_{k=0}^{p-1} (1+kx) = (1 - x^{p-1}).$$

Thus $c_p(i_*(\rho)) \neq 0$ while $c_i(\rho) = 0$ for all $i > 0$.

If a formula for $c_i(i_*(\rho))$ were known, for ρ 1-dimensional, then one could use Brauer's Theorem [5] and (3) to determine all Chern classes.

Relation with the spectral sequence.

We shall now describe the relation between the Chern classes of a representation and our spectral sequence. The statements which follow are given without proof, but they are all elementary consequences of the results in [1].

We recall that R(G) is filtered by subgroups $R_{2n}(G)$ defined topologically. If we filter the group A(G) by defining $A_{2n}(G)$ to be the subgroup of elements $a = \prod_{\nu=0}^{\infty} a_{2\nu}$ with $a_{2\nu} = 0$ for $1 \leqslant \nu \leqslant n-1$, then

(9) $c : R(G) \to A(G)$ *is a homomorphism of filtered groups.*

In particular c induces a homomorphism of completions (A(G) is itself complete)

$$\hat{c} : \widehat{R(G)} \to A(G),$$

and $c(\sigma) = 0$ for σ in the kernel of $R(G) \to \widehat{R(G)}$ as stated above.

Let $H'(G, Z) \subset H^*(G, Z)$ denote the subgroup of "universal cycles" in the spectral sequence $H^*(G, Z) \Rightarrow \widehat{R(G)}$, i.e. $H'(G, Z) = Z_\infty$ in the notation of § 3. From the spectral sequence we obtain an epimorphism

$$\varphi : H'(G, Z) \to GR(G).$$

Then we have:

(10) *For all* $\rho \in R(G)$ *and all* i $\qquad c_i(\rho) \in H'(G, \mathbf{Z})$,

(11) *Let* $\rho \in R_{2n}(G)$, $[\rho]$ *the image of* ρ in $G^{2n}R(G)$. *Then*

$$\varphi(c_n(\rho)) = (-1)^{n-1}(n-1)!\,[\rho].$$

There is also a close relation between Chern classes and the operators γ^n of § 12. If $\rho \in R(G)$ then, by definition of $R_{2n}(G)$, $\qquad \gamma^n(\rho - \varepsilon(\rho)) \in R'_{2n}(G).$

Since $R'_{2n}(G) \subset R_{2n}(G)$ (12.6) we obtain an element $[\gamma^n(\rho - \varepsilon(\rho))] \in G^{2n}R(G)$. Then

(12) *For any* $\rho \in R(G)$ *we have* $\qquad \varphi(c_n(\rho)) = [\gamma^n(\rho - \varepsilon(\rho))].$

From this we see that the conjecture that $R'_{2n}(G) = R_{2n}(G)$ which was made in § 12 is equivalent to the following conjecture: the subring of $H'(G, Z)$ generated by all Chern classes is mapped, by φ, *onto* GR(G).

We already know (2.5) that c_1 induces an isomorphism $G^2R(G) \to H^2(G, \mathbf{Z})$. It follows from (11) that c_2 induces a *monomorphism* (¹) $\qquad G^4R(G) \to H^4(G, \mathbf{Z}).$

Thus up to this dimension the filtration is determined by the Chern classes. This is no longer true in higher dimensions as is shown by the example above with G cyclic of order p^2.

In conclusion we may add that for *real* representations ρ one can introduce Stiefel-Whitney classes $w_i(\rho) \in H^i(G, Z_2)$ [2, § 10]. Their formal properties are similar to those of Chern classes.

(¹) From the spectral sequence view-point this corresponds to the fact that the first non-zero group B^p_∞ arises for $p=6$.

REFERENCES

[1] M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Differential Geometry, *Proc. of Symp. in Pure Math.*, vol. 3, Amer. Math. Soc., 1961.
[2] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces I, *Amer. J. Math.*, 80 (1958), pp. 458-538.
[3] R. Bott, The space of loops on a Lie group, *Mich. Math. J.*, 5 (1958), pp. 35-61.
[4] R. Bott, Some remarks on the periodicity theorems, Colloque de Topologie, Lille, 1959, *Bull. Soc. Math. France*, 87 (1959), pp. 293-310.
[5] R. Brauer and J. Tate, On the characters of finite groups, *Ann. of Math.*, 62 (1955), pp. 1-7.
[6] H. Cartan and C. Chevalley, *Séminaire E.N.S.*, Paris, 1956.
[7] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton, University Press, 1956.
[8] J. Dieudonné and A. Grothendieck, Éléments de Géométrie algébrique, *Publ. Math.*, Inst. Hautes Études Scient. (to appear).
[9] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Springer, Berlin, 1956.
[10] J. Milnor, Construction of universal bundles II, *Ann. of Math.*, 63 (1956), pp. 430-436.
[11] D. G. Northcott, *Ideal Theory*, Cambridge University Press, 1953.
[12] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1951.
[13] J. H. C. Whitehead, Combinatorial homotopy I, *Bull. Amer. Math. Soc.*, 55 (1949), pp. 213-245.
[14] O. Zariski and P. Samuel, *Commutative Algebra* (vol. I), Van Nostrand, Princeton, 1958.

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